Digital Signal Processing Summer Seminar
Institute of Information Technology - Vietnam Education Foundation

FUNDAMENTALS OF SIGNAL PROCESSING

Problem Set 2 - Sampling and Frequency Analysis
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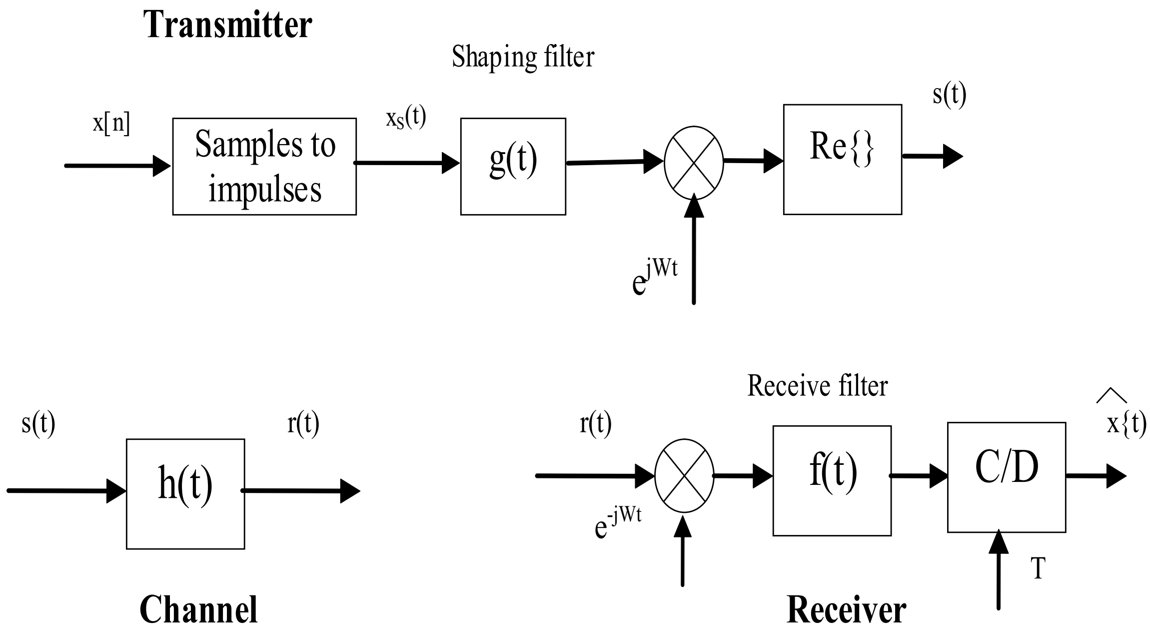
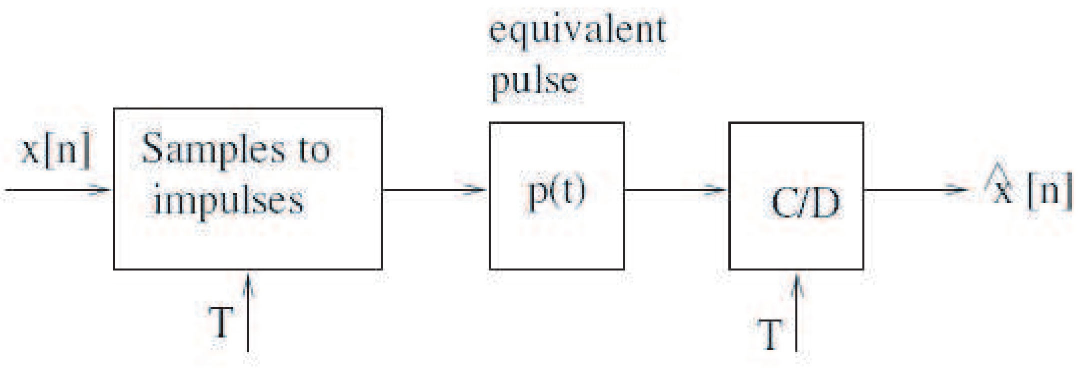
1. The following figure depicts a passband communication system using pulse-amplitude modulation, where the information sequence is given by $x[n]$. The transmitter of the system converts the information sequence to a continuous-time signal using the pulse-shaping filter $g(t)$ and then modulates the signal up to a center frequency of W to produce the transit waveform $s(t)$; The continuous time channel has impulse response $h(t)$. At the receiver, the passband signal is then modulated back to baseband, filtered with the receive filter with impulse response $f(t)$, and then sampled with an ideal C/D converter.



(a) Asssuming that $g(t)$, $h(t)$ and $f(t)$ are sufficiently bandlimited, and that W is appropriately chosen, the overall system above from input sequence to output sequence can be represented through an equivalent baseband pulse model as shown below. Determine the precise conditions on the center frequency W and the bandwidths of $G(j\Omega)$, $H(j\Omega)$, and $F(j\Omega)$, in terms of W and the sampling time T such that this equivalent model holds for some $p(t)$.

(b) Under the conditions specified in part (a), determine $p(t)$. You may express your answer in terms of either the time-domain signals $g(t)$, $h(t)$ and $f(t)$, or their Fourier transforms, $G(j\Omega)$, $H(j\Omega)$, and $F(j\Omega)$, and the center frequency W.

(c) Assume the conditions from part (a) are satisfied, and the equivalent pulse model is valid. Under what conditions on the pulse shape $p(t)$ will the overall system from input



$x[n]$ to output $\hat{x}[n]$ behave like an LTI system, i.e., where

$$\hat{x}[n] = \sum_{k=-\infty}^{\infty} x[k] h_{eq}[n-k]$$

Also determine $h_{eq}[n]$ under these conditions.

2. *Sampling*

   (a) Let $x[n]$ be a discrete-time signal whose discrete-time Fourier transform $X_d(\omega)$ exists. Explain why for every positive T, there exists a *unique* continuous-time signal $x_c(t)$ such that its Fourier transform $X_c(\omega)$ is bandlimited to $|\omega| \leq \pi/T$ and $x[n] = x_c(nT)$, $\forall n \in \mathbb{Z}$.

   (b) Given a continuous-time signal $x(t)$ such that its Fourier transform $X(\omega)$ is bandlimited to $|\omega| \leq \pi/T$, find a formula that recovers $x(t)$ from the average samples

$$\tilde{x}[n] = \int_{nT}^{(n+1)T} x(t)dt, \quad n \in \mathbb{Z}.$$

3. *Interpolation*
   A function $f(t)$ is called an *interpolation* function if

$$f(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for all other integers } n \end{cases}$$

   (a) Show that $f(t)$ is an interpolation function if and only if

$$\sum_{k=-\infty}^{\infty} F(\omega + 2k\pi) = 1,$$

   where $F(\omega)$ is the Fourier transform of $f(t)$.

   (b) Suppose that $f(t)$ is obtained from a D-to-A (digital-to-analog) converter using a discrete-time sequence $h[n]$ and a kernel function $\phi(t)$

$$f(t) = \sum_{n=-\infty}^{\infty} h[n]\ \phi(t-n)$$

   Find the DTFT (discrete-time Fourier transform) $H(e^{j\omega})$ of $h[n]$ in term of the Fourier transform $\Phi(\omega)$ of $\phi(t)$ such that $f(t)$ is an interpolation function.

(c) Now suppose that instead $f(t)$ is obtained from a D-to-A converter of double rate

$$f(t) = \sum_{n=-\infty}^{\infty} h[n]\ \phi(2t-n),$$

and also suppose that the kernel $\phi(t)$ is an interpolation function. Find the condition on $H(e^{j\omega})$ such that $f(t)$ is an interpolation function.

4. *Frequency analysis*
Imagine that you have collected $N = 140$ equally spaced data values over a time interval of duration $A = 14$ seconds. You perform spectral analysis of this data set using the DFT of length 140. What are the minimum (non-zero) frequency f_1 and maximum frequency f_{70} that are returned by the DFT coefficients of this data set? Give your answers in Hertz.

5. *The Short-Time Fourier Transform (STFT)*
Short-Time Fourier Transform (STFT) measures the spectral characteristics of a time-varying signal, as a function of both time and frequency. Specifically, the STFT at a time n and at frequency $\omega_k = \frac{2\pi k}{N}$ is defined to be

$$X(k,n) = \sum_{m=-\infty}^{\infty} x[m]w[n-m]e^{j\frac{2\pi k(n-m)}{N}},\ 0 \le k \le N-1$$

where $x[n]$ is the signal of interest, and $w[n]$ is a window function.

(a) The STFT can be written as the N-point DFT of some signal $x_n[m]$. Find $x_n[m]$ as a function of $x[m]$ and $w[m]$. Be sure to consider the possibility that the window $w[n]$ may be longer than N samples.

(b) In most applications, the most efficient method for computing the STFT is to shift $x[n]$, window and take the FFT. If only a few frequency samples are desired, however, it maybe more efficient to compute the STFT using a bank of filter. Find the impulse response $h_k[n]$ of some filter $H_k(z)$ such that $X(k,n) = x[n] * h_k[n]$.

(c) The STFT is usually computed using a finite length window $w[n]$, but in some applications, it is useful to use an infinite length window. Consider in particular the window $w_{IIR}[n] = a^n u[n]$ for real-valued forgetting factor a, $0 < a < 1$. Show that, using $w_{IIR}[n]$, the filter $H_k(z)$ may be implemented using a second order IIR system with real denominator coefficients. Draw a second-order direct form II implementation of $H_k(z)$ with real-valued feedback coefficients.

6. *Discrete Cosine Transform*
The MPEG audio coder computes the DCT (more precisely DCT-IV) of each signal block of length N, $x[n]_{n=0}^{N-1}$, defined as follows

$$X_{DCT}[k] = \sum_{n=0}^{N-1} x[n] \cos\left(\frac{\pi}{N}\left(k+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\right),\ 0 \le k \le N-1$$

(a) Show that it is possible to write $X_{DCT}[k]$ in terms of $X(e^{jw})$, the DTFT of $x[n]$ as

$$X_{DCT}[k] = \frac{1}{2}\left(G(e^{jw_k})X(e^{jw_k}) + G(e^{-jw_k})X(e^{-jw_k})\right)$$

for some frequency samples w_k and some frequency-domain-twiddle-factor $G(e^{jw})$. Find w_k and $G(e^{jw})$. Plot the locations of the frequency samples w_k for $N = 4$.

(b) Consider the problem of trying to compute $X(e^{jw_k})$ at the frequency samples w_k specified in part (a). Show that $X(e^{jw_k}) = Y[k]$, where $Y[k]$ is the $2N$-point DFT of some sequence $y[n] = f[n]x[n]$, and $f[n]$ is some time-domain twiddle factor. Find $f[n]$.