Digital Signal Processing Summer Seminar
Institute of Information Technology - Vietnam Education Foundation

FUNDAMENTALS OF SIGNAL PROCESSING

Problem Set 4 - Statistical and Adaptive Signal Processing

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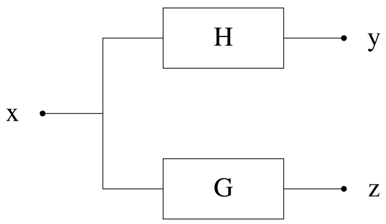
1. *Random signals*
   Consider the following system where $x[n]$ is a wide-sense stationary random signal and $H(z)$ and $G(z)$ are two fixed stable filters. Consider the autocorrelation function of $x[n]$, and the



   cross-correlation function of $y[n]$ and $z[n]$

$$\gamma_{xx}[m] = E\{x[n]x^*[n-m]\}$$
$$\gamma_{yz}[m] = E\{y[n]z^*[n-m]\}.$$

   Show that in the Fourier domain we have:

$$\Gamma_{yz}(e^{j\omega}) = \Gamma_{xx}(e^{j\omega})H(e^{j\omega})G^*(e^{j\omega}).$$

2. *Wiener filter*
   Consider the following filtering system:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n],$$

   where input $\{x[n]\}$ is a wide-sense stationary process with $\gamma_{xx}[m] = E\{x[n]x^*[n-m]\}$, and $\{h[n]\}_{n=0}^{K-1}$ is a fixed filter.

   (a) Let $\gamma_{yx}[m] = E\{y[n]x^*[n-m]\}$ be the cross-correlation sequence between $x[n]$ and $y[n]$. Show that

$$\Gamma_{yx}(z) = H(z)\Gamma_{xx}(z).$$

   (b) Develop a length-L Wiener (or MMSE) filter $\{g[n]\}_{n=0}^{L-1}$ to estimate $\{y[n]\}$ from $\{x[n]\}$:

$$\hat{y}[n] = \sum_{l=0}^{L-1} g[l]x[n-l],$$

   that minimize the MSE $E\{|y[n] - \hat{y}[n]|^2\}$. In particular, write down a set of normal equations to solve $\{g[n]\}_{n=0}^{L-1}$ from $\{h[n]\}_{n=0}^{K-1}$ and $\{\gamma_{xx}[m]\}$.

   (c) For $K = L + 1$, show that the Wiener filter can be written as

$$g[k] = h[k] + h[L]b[k], \quad k = 0, 1, \ldots, L-1.$$

   where $\{b[k]\}_{k=0}^{L-1}$ is the optimal order-L linear backward predictor for $\{x[n]\}$.

3. Consider an estimation problem, where a signal $x[n]$ is to be represented by a constant c. Let the autocorrelation function of $x[n]$ be given by $E\{x[n]x[n-m]\} = \gamma_{xx}[m]$.

   (a) Determine the value of c which minimizes the mean-squared error, $\epsilon = E\left\{(x[n]-c)^2\right\}$.

   (b) Suppose that an adaptive algorithm is to be used to adaptively estimate c from the data $x[n]$. Let $c[n]$ be the current value of the estimate. Find the gradient of the mean squared error with respect to the coefficient c. By analogy to the LMS algorithm, derive a "stochastic gradient" update algorithm for $c[n]$. Note that like the LMS algorithm, the update must not require knowledge of the statistics of $x[n]$.

4. For the same estimation problem given in Problem 3,

   (a) If $E\{c[n]\}$ converges, to what does it converge? Make appropriate assumptions (e.g. you may use independence theory).

   (b) Under what conditions does this algorithm converge in the mean, i.e., $E\{c[n]\} \rightarrow \bar{c}^*$ for some $\bar{c}^*$? Make appropriate assumptions (e.g. you may use independence theory).

5. *Leaky-LMS algorithm*
   The Leaky-LMS algorithm is an adaptation strategy which attempts to minimize the following cost function:

$$\epsilon_L = E\left\{e^2[n] + \lambda \bar{w}[n]^T \bar{w}[n]\right\}$$

   $\bar{w}$ where $\lambda > 0$ and $\bar{w}[n]$ are the coefficients of the usual adaptive filter problem, i.e.,

$$\begin{aligned} \hat{y}[n] &= \bar{w}[n]^T \boldsymbol{x}[n] \\ \bar{x}[n] &= [x[n], x[n-1], ..., x[n-p+1]]^T \\ e[n] &= d[n] - \hat{y}[n]; \end{aligned}$$

   where $d[n]$ is the desired response, and $\bar{x}[n]$ is the vector of inputs to the filter. Assume that $x[n]$ and $d[n]$ are zero-mean, wide-sense stationary random processes. The autocorrelation function of $x[n]$ is given by $E\{x[n]x[n-m]\} = \gamma_{xx}[m]$, and the cross correlation between $d[n]$ and $x[n]$ is given by $E\{d[n]x[n-m]\} = \gamma_{dx}[m]$.

   (a) Determine $\bar{w}_{opt}$, the set of filter coefficients which minimize ϵ_L. Clearly define all terms in your expression.

   (b) Determine $\nabla_{\bar{w}}\epsilon_L$, the gradient of the error with respect to the filter coefficients. By analogy to the LMS algorithm, determine the leaky-LMS update equations. Note that as with the LMS algorithm, this should not depend on knowledge of $\Gamma_{xx}[m]$ or $\gamma_{dx}[m]$.

   (c) Determine $\epsilon_L^* = \min_{\bar{w}} E\left\{e^2[n] + \lambda \bar{w}^T \bar{w}\right\}$

6. Using the same Leaky-LMS algorithm given in Problem 5,

   (a) If $E\{\bar{w}[n]\}$ converges, to what does it converge? Make appropriate assumptions (e.g. you may use independence theory).

   (b) Under what conditions does the leaky-LMS algorithm converge in the mean, i.e., $E\{\bar{w}[n]\} \rightarrow \bar{w}^*$ for some $\bar{w}^*$? Make appropriate assumptions (e.g. you may use independence theory).