Digital Signal Processing Summer Seminar
Institute of Information Technology - Vietnam Education Foundation
Image/Video Coding and Processing

Homework Assignment I

July 2006

1. Let X denote a random variable distributed on the set $\{x_1, x_2, \ldots, x_N\}$ with associated probabilities $\{P_1, P_2, \ldots, P_N\}$. Let Y be another random variable defined on the same set but distributed uniformly.

   (a) Show that
   $$H(X) \ \leq \ H(Y).$$
   Hint : First prove the inequality $\ln w \leq w - 1$ with equality for $w = 1$, then apply this inequality to $\sum_{n=1}^{N} P_n \ln \frac{1/N}{P_n}$.

   (b) Show that the entropy H of $\mathcal{X}$ satisfies
   $$0 \ \leq \ H \ \leq \ \log N.$$

   (c) Find the necessary and sufficient conditions under which equality holds.

   (d) What do the results in (a), (b) and (c) tell us? Explain the significance of this problem.

2. Consider a source with 3 symbols and corresponding probabilities 0.95, 0.04, and 0.01.

   (a) Find the entropy of this source.

   (b) Design the Huffman code for this source. Compare your average codeword length with the entropy.

   (c) Combine two symbols and re-design your Huffman code. Again, compare your average codeword length to the entropy. Is there any improvement?

3. *Chain Rule for Entropy:*
   The Joint Entropy of a pair of random variables is defined as
   $$H(X, Y) = -E_{p(x,y)} \log p(x, y) = -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(X, Y),$$
   where $p(x, y)$ is the joint distribution (pdf) of the 2 random variables.
   The Conditional Entropy of a random variable given another random variable is defined as the expected value of the conditional distribution, averaged over the conditioning random variable. Mathematically,
   $$H(Y|X) = -E_{p(x,y)} \log p(y|x) = -\sum_{x \in \mathcal{X}} p(x) \log p(Y|X)$$
   Prove that:

   (a) $H(X, Y) = H(X) + H(Y|X)$

   (b) $H(X, Y|Z) = H(X|Z) + H(Y|X, Z)$

   (c) $H(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} H(X_i|X_{i-1}, \ldots, X_1)$