Course 18.327 and 1.130
Wavelets and Filter Banks

Sampling rate change operations: upsampling and downsampling; fractional sampling; interpolation

Downsampling

Definition:

$$(\downarrow 2)\begin{bmatrix} \vdots \\ x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ x[0] \\ x[2] \\ x[4] \\ \vdots \\ \& \\ \end{bmatrix}$$

As a matrix operation:

$$\begin{bmatrix} & & & \vdots & & & \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & & & \vdots & & & \end{bmatrix} \begin{bmatrix} \vdots \\ x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ \vdots \end{bmatrix} = \begin{bmatrix} : \\ x[0] \\ x[2] \\ x[4] \\ : \\ : \end{bmatrix}$$

Upsampling

Definition:

$$(\uparrow 2)\begin{bmatrix}\vdots\\ x[0]\\ x[1]\\ x[2]\\ \vdots\end{bmatrix} = \begin{bmatrix}\vdots\\ x[0]\\ 0\\ x[1]\\ 0\\ x[2]\\ 0\\ \vdots\end{bmatrix}$$

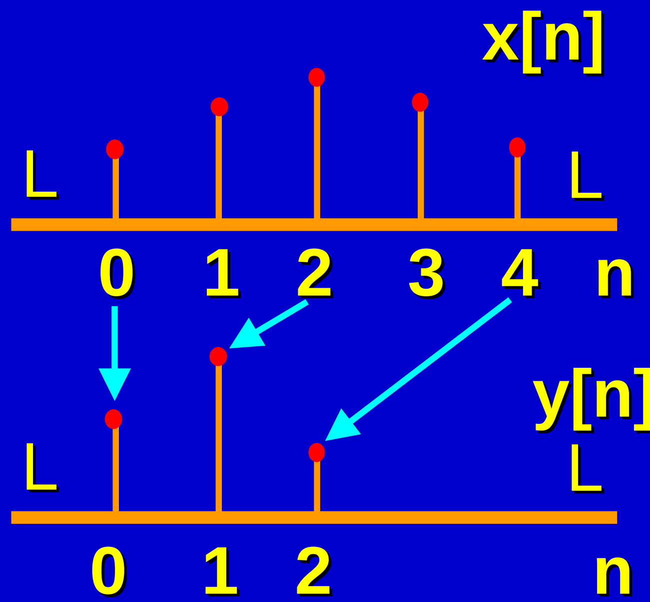
As a matrix operation:

$$
\begin{bmatrix}
 & & \mathrm{M} & & \\
\mathrm{L} & 1 & 0 & 0 & \mathrm{L} \\
\mathrm{L} & 0 & 0 & 0 & \mathrm{L} \\
\mathrm{L} & 0 & 1 & 0 & \mathrm{L} \\
\mathrm{L} & 0 & 0 & 0 & \mathrm{L} \\
\mathrm{L} & 0 & 0 & 1 & \mathrm{L} \\
\mathrm{L} & 0 & 0 & 0 & \mathrm{L} \\
 & & \mathrm{M} & &
\end{bmatrix}
\begin{bmatrix}
\mathrm{M} \\ x[0] \\ x[1] \\ x[2] \\ : \\ : \\ :
\end{bmatrix}
=
\begin{bmatrix}
\mathrm{M} \\ x[0] \\ 0 \\ x[1] \\ 0 \\ x[2] \\ 0 \\ \mathrm{M}
\end{bmatrix}
$$

Downsampling

Downsampling by 2

$$x[n] \longrightarrow \boxed{\;\downarrow 2\;} \longrightarrow y[n]$$



$$y[n] = x[2n]$$

$$Y(\omega) = \sum_{n} x[2n]e^{-i\omega n}$$

$$= \sum_{m\ \text{even}} x[m]e^{-i\omega m/2}$$

$$= \tfrac{1}{2} \sum_{m} \{1 + (-1)^m\}\, x[m]e^{-i\omega m/2}$$

$$= \tfrac{1}{2} \Big\{\sum_{m} x[m]e^{-i\omega \frac{m}{2}} + \sum_{m} x[m]e^{-i(\frac{\omega}{2} + \pi)m}\Big\} ;$$

$$(-1)^m = e^{-i\pi m}$$

$$= \tfrac{1}{2} \{X(\omega/2) + X(\omega/2 + \pi)\}$$

Downsampling by M

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow y[n]$$

$$y[n] \quad = \quad x[Mn]$$

$$Y(\omega) \;=\; \sum_{m\,=\,nM} x[m]e^{-i\omega m/M}$$

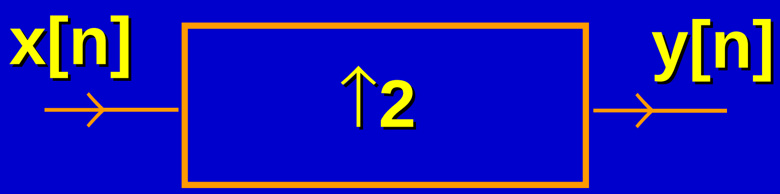
$$= \frac{1}{M} \sum_{m} \{ \sum_{k\,=\,0}^{M-1} e^{-i\frac{2\pi}{M}km} \}\, x[m]e^{-i\omega m/M} \; ;$$

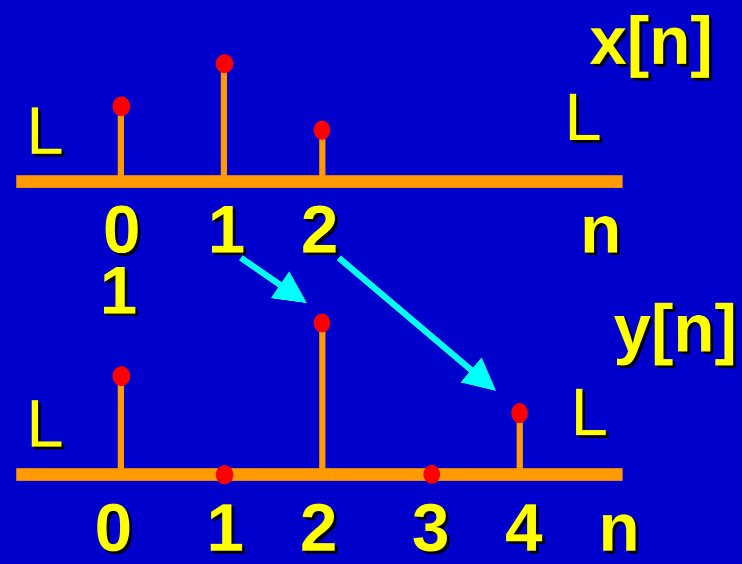
$$\frac{1}{M} \sum_{k\,=\,0}^{M-1} (e^{-i\frac{2\pi}{M}m})^{k} = \begin{cases} 1 \text{ if } m = nM \\ 0 \text{ if } m \neq nM \end{cases}$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} X\!\left(\frac{\omega + 2\pi k}{M}\right)$$

Upsampling

Upsampling by 2





$$y[n] = \begin{cases} x[n/2] & ; \text{ n even} \\ 0 & ; \text{ n odd} \end{cases}$$

$$Y(\omega) = \sum_{n \text{ even}} x[n/2]e^{-i\omega n}$$

$$= \sum_{m} x[m]e^{-i\omega 2m}$$

$$= X(2\omega)$$

Upsampling by L

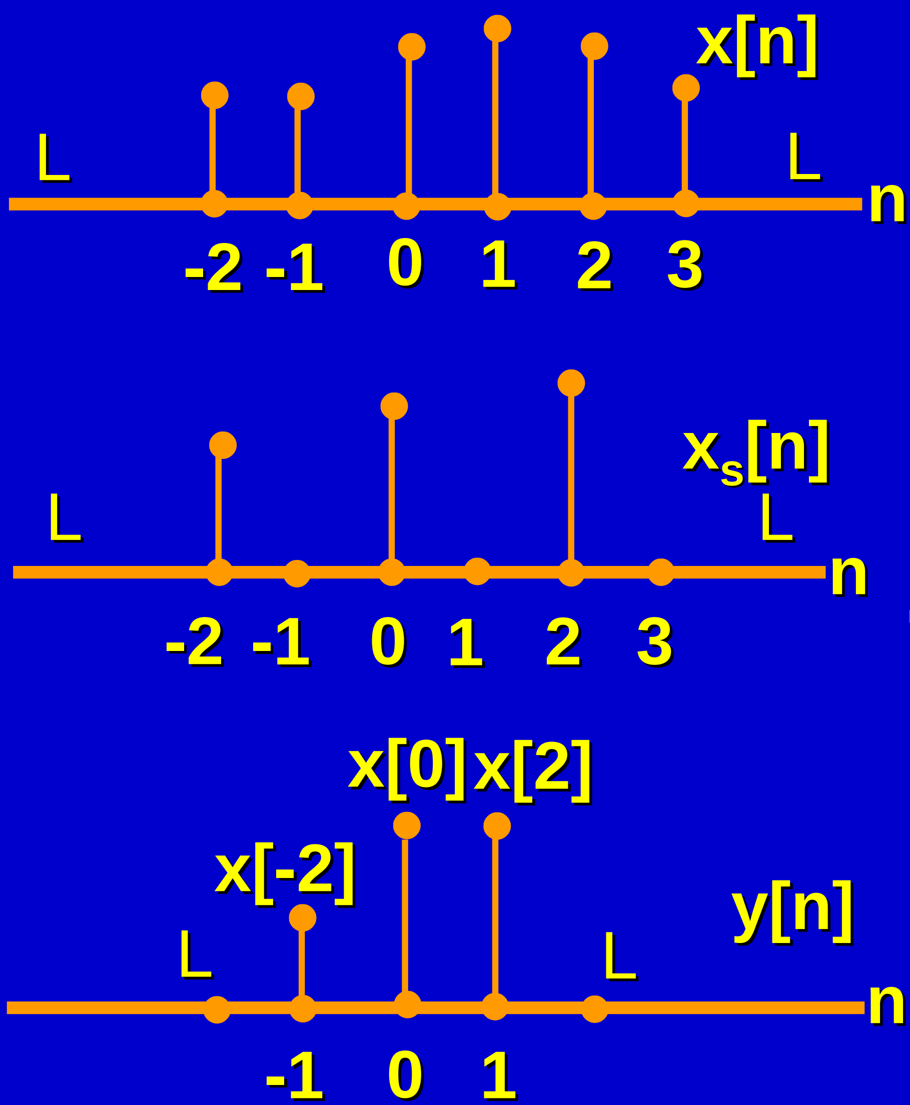
$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow y[n]$$

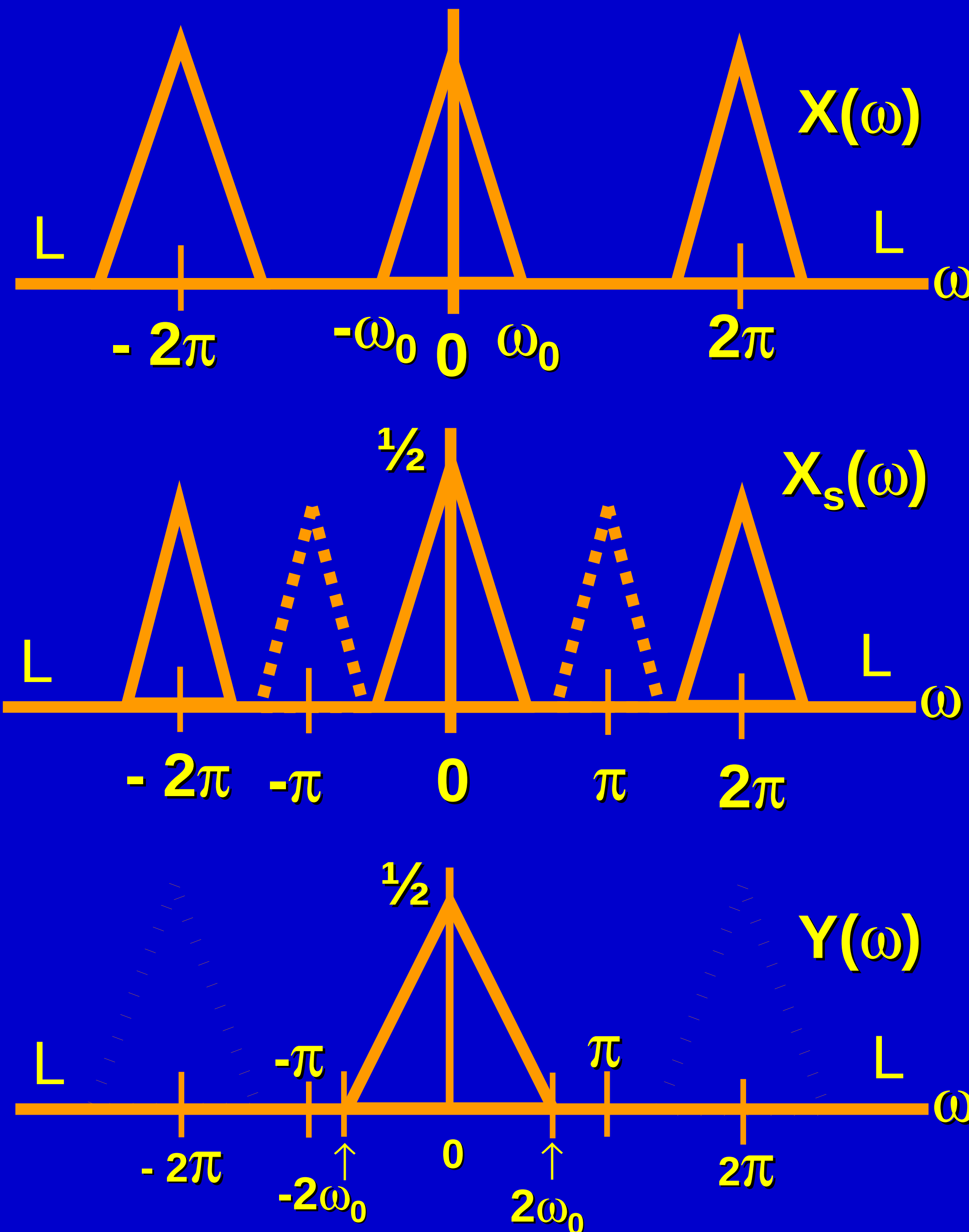
$$y[n] = \begin{cases} x[n/L] & ; \ n = mL \\ 0 & ; \ n \neq mL \end{cases}$$

$$Y(\omega) = \sum_{n=mL} x[n/L]\, e^{-i\omega n}$$

$$= X(L\omega)$$

Downsampling

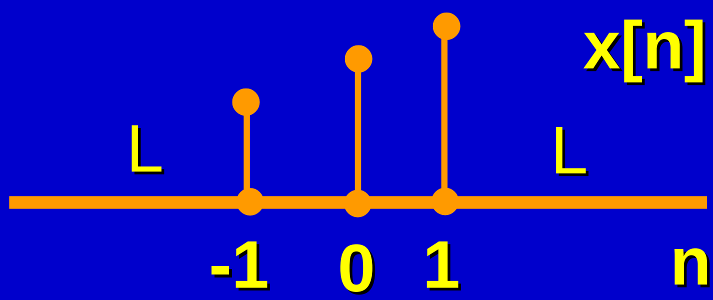


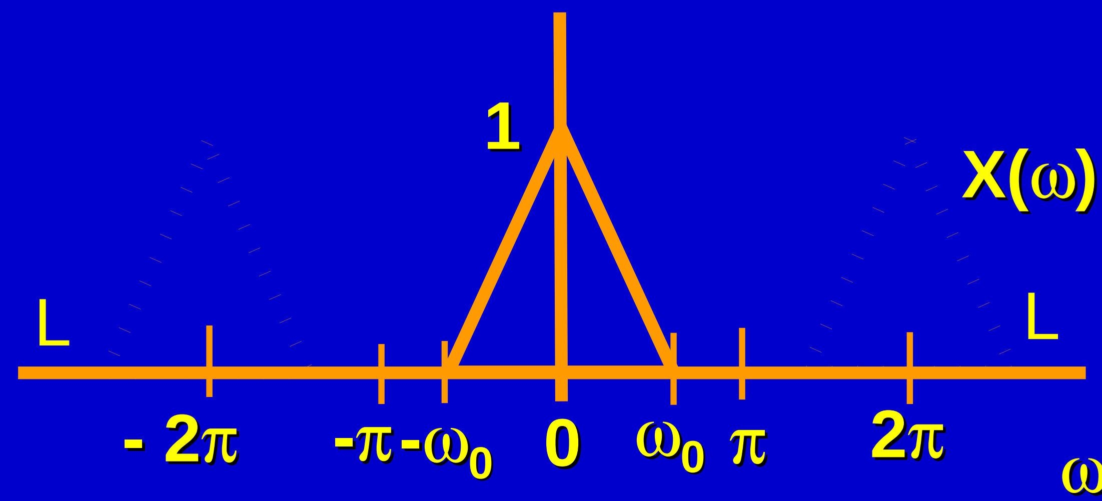


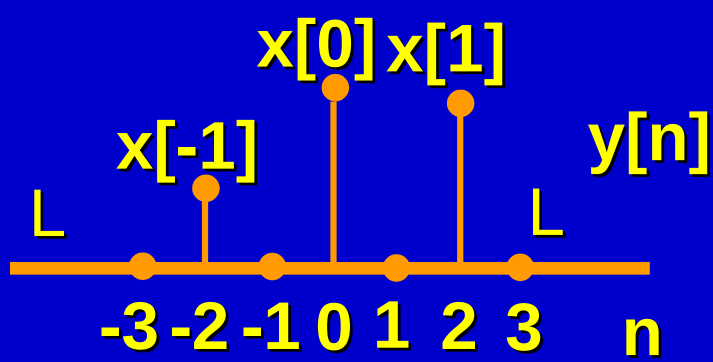
$$y[n] = (\downarrow 2)\, x[n] = x[2n]$$

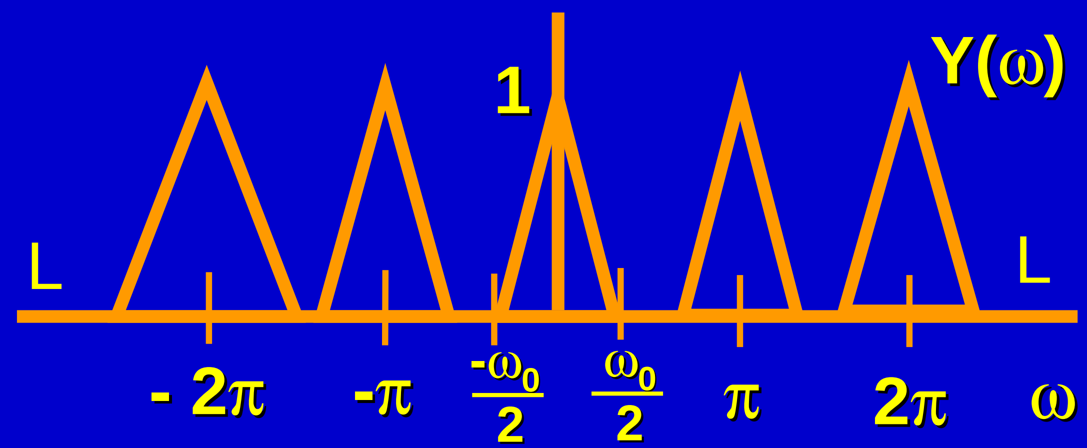
$$Y(\omega) = \tfrac{1}{2}\left\{ X\!\left(\frac{\omega}{2}\right) + X\!\left(\frac{\omega}{2} + \pi\right)\right\}$$

Upsampling







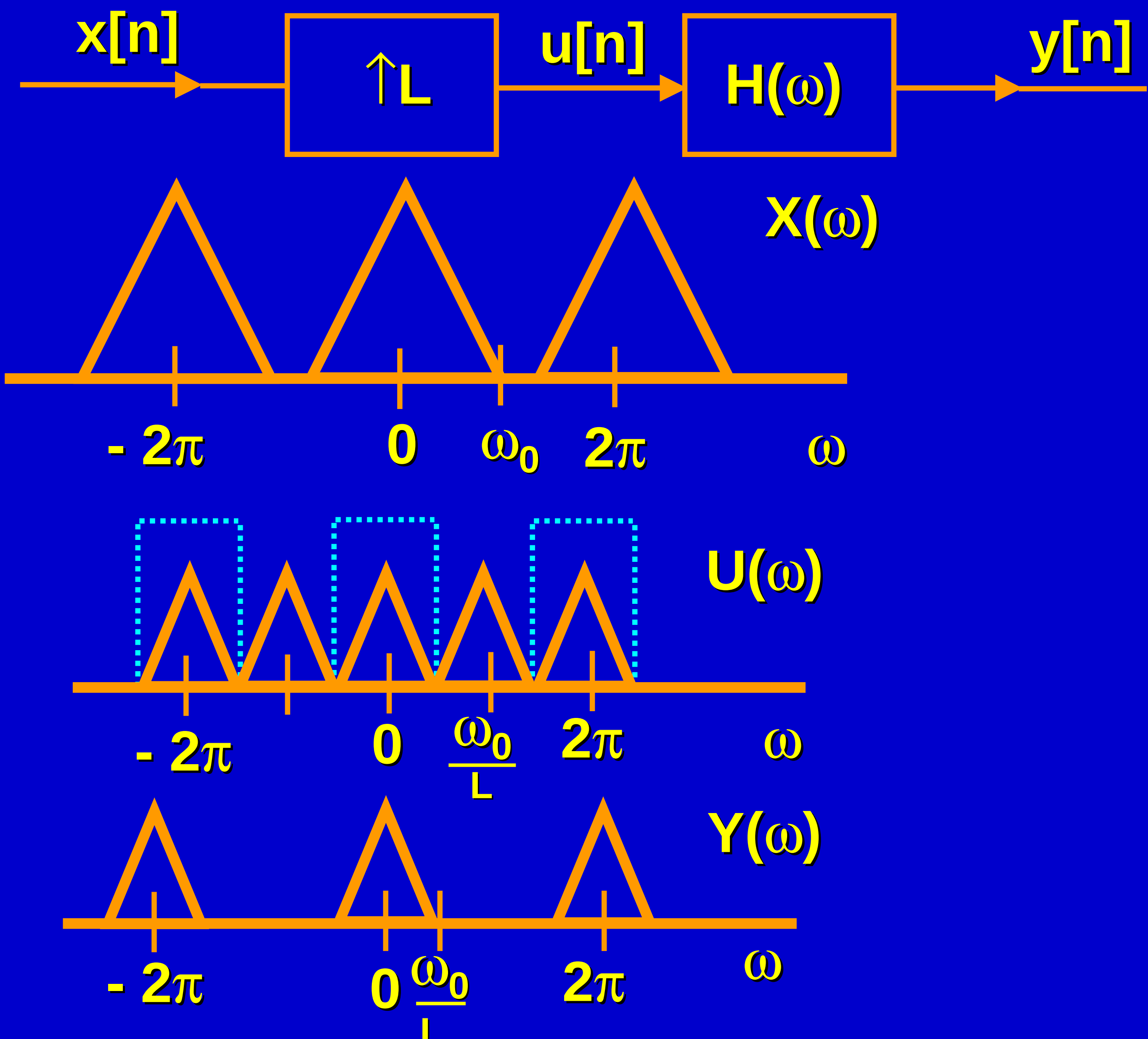


$$y[n] = \begin{cases} x[n/2] & ; \text{n even} \\ 0 & ; \text{n odd} \end{cases}$$

$$Y(\omega) = X(2\omega)$$

Interpolation

Use lowpass filter after upsampling



Fractional Sampling

Consider

$$x[n] \longrightarrow \boxed{\uparrow L} \xrightarrow{u[n]} \boxed{\downarrow M} \longrightarrow y[n]$$

$$Y(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} U\left(\frac{\omega + 2\pi k}{M}\right)$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\omega + 2\pi k}{M} L\right)$$

What about

$$x[n] \longrightarrow \boxed{\downarrow M} \xrightarrow{d[n]} \boxed{\uparrow L} \longrightarrow y[n]$$

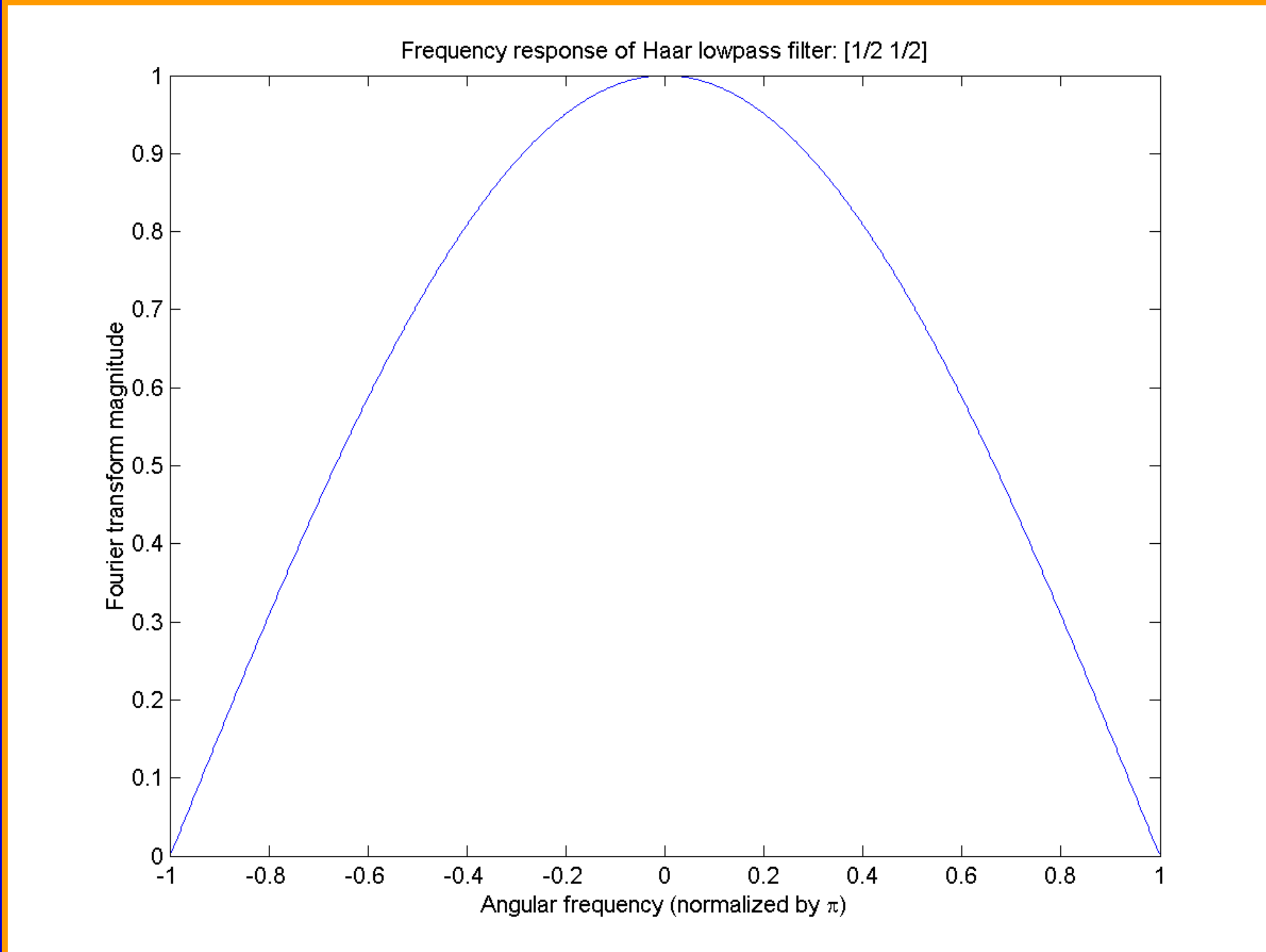
$$Y(\omega) = D(\omega L)$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\omega L + 2\pi k}{M}\right)$$

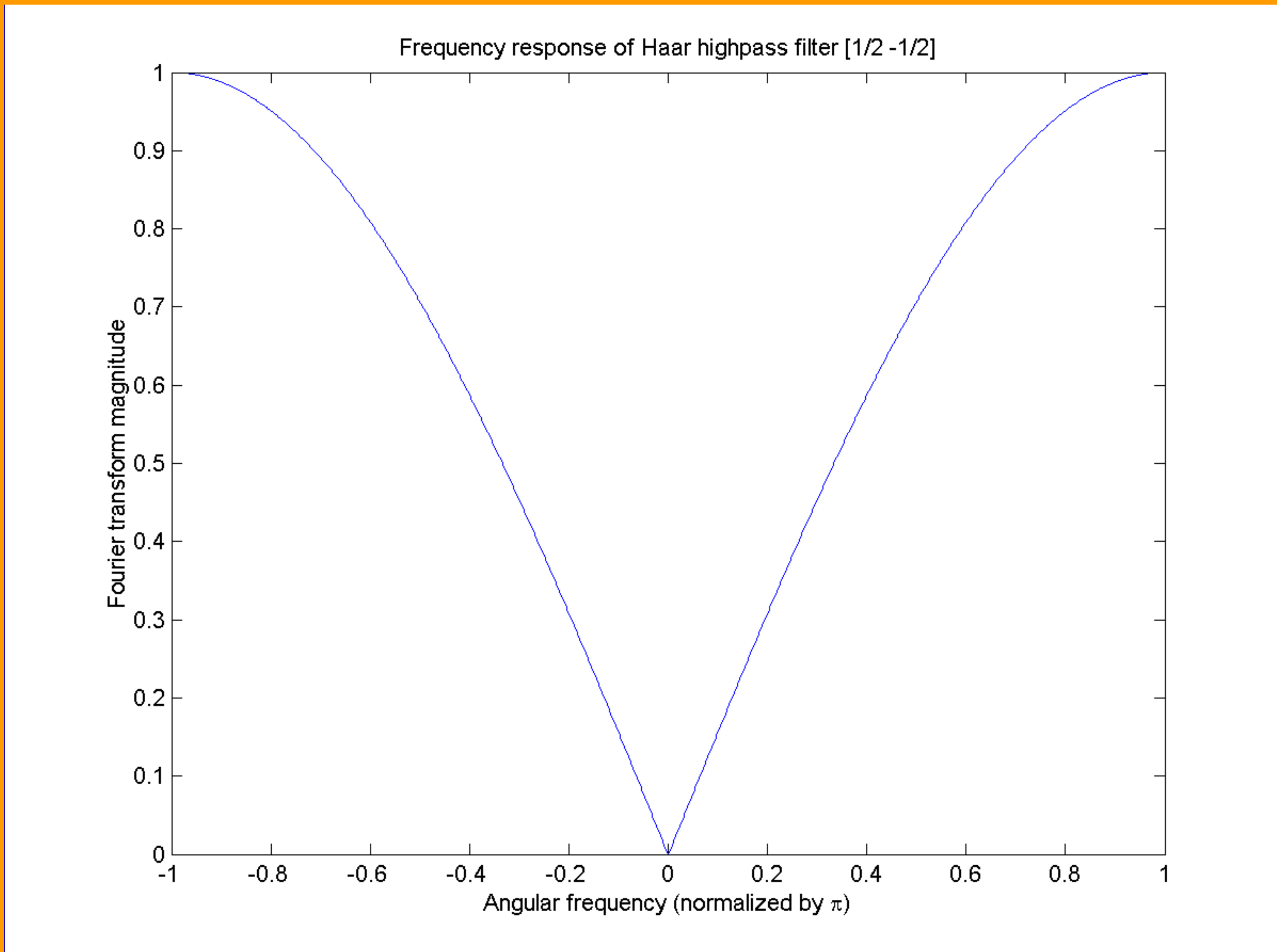
Matlab Example 1

Basic filters, upsampling and downsampling.

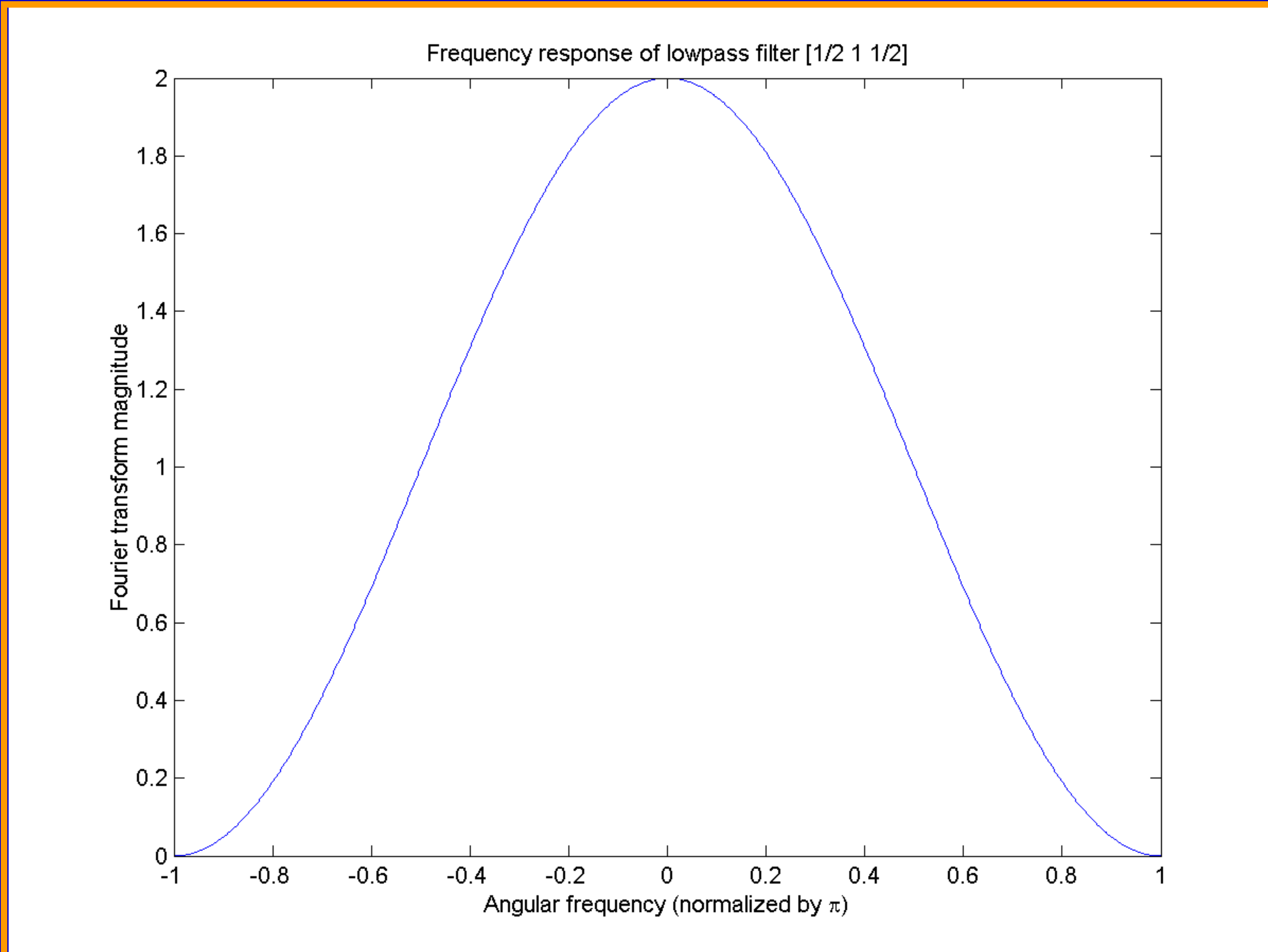
Lowpass filter



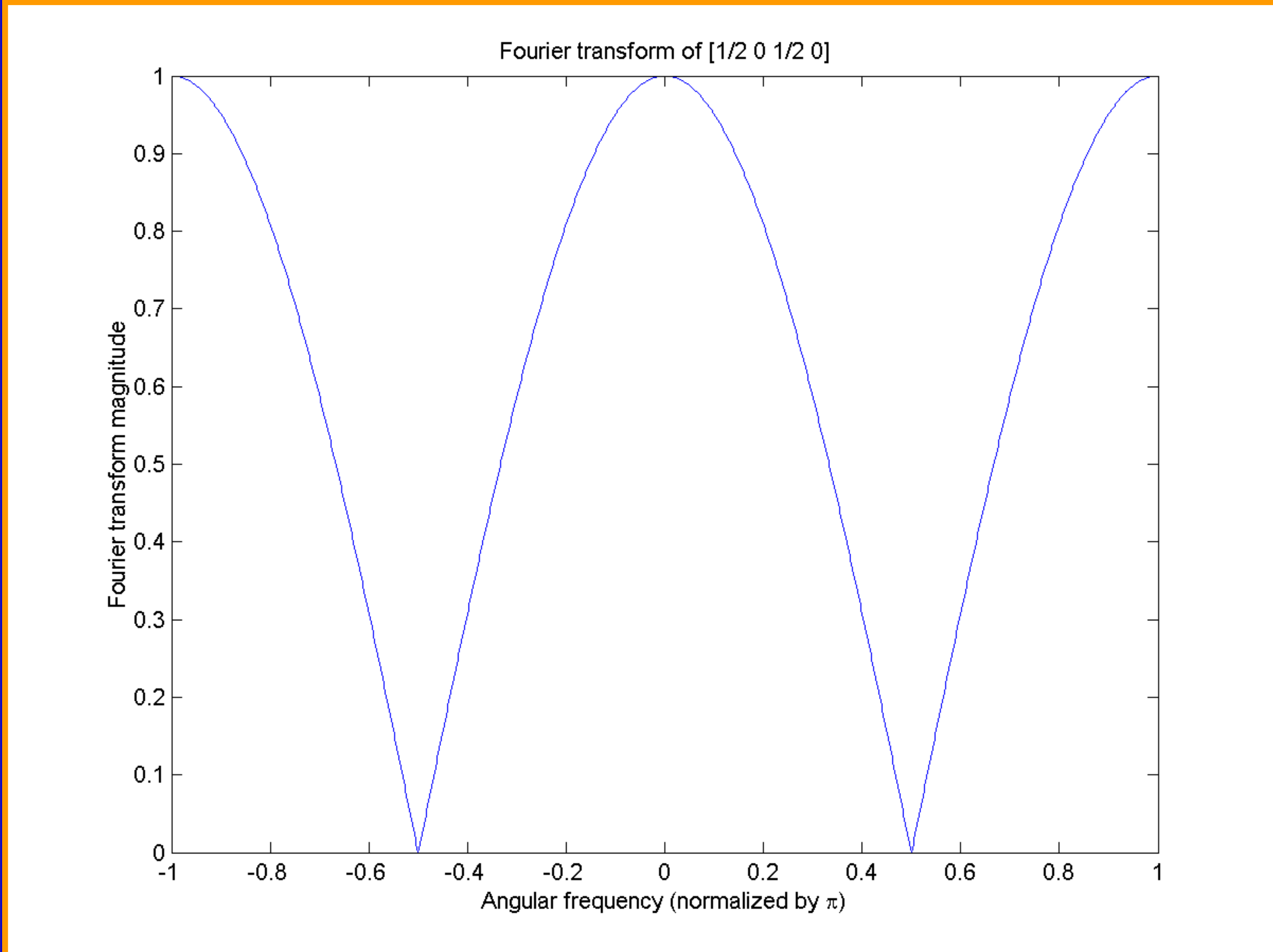
Highpass filter



Linear interpolating lowpass filter

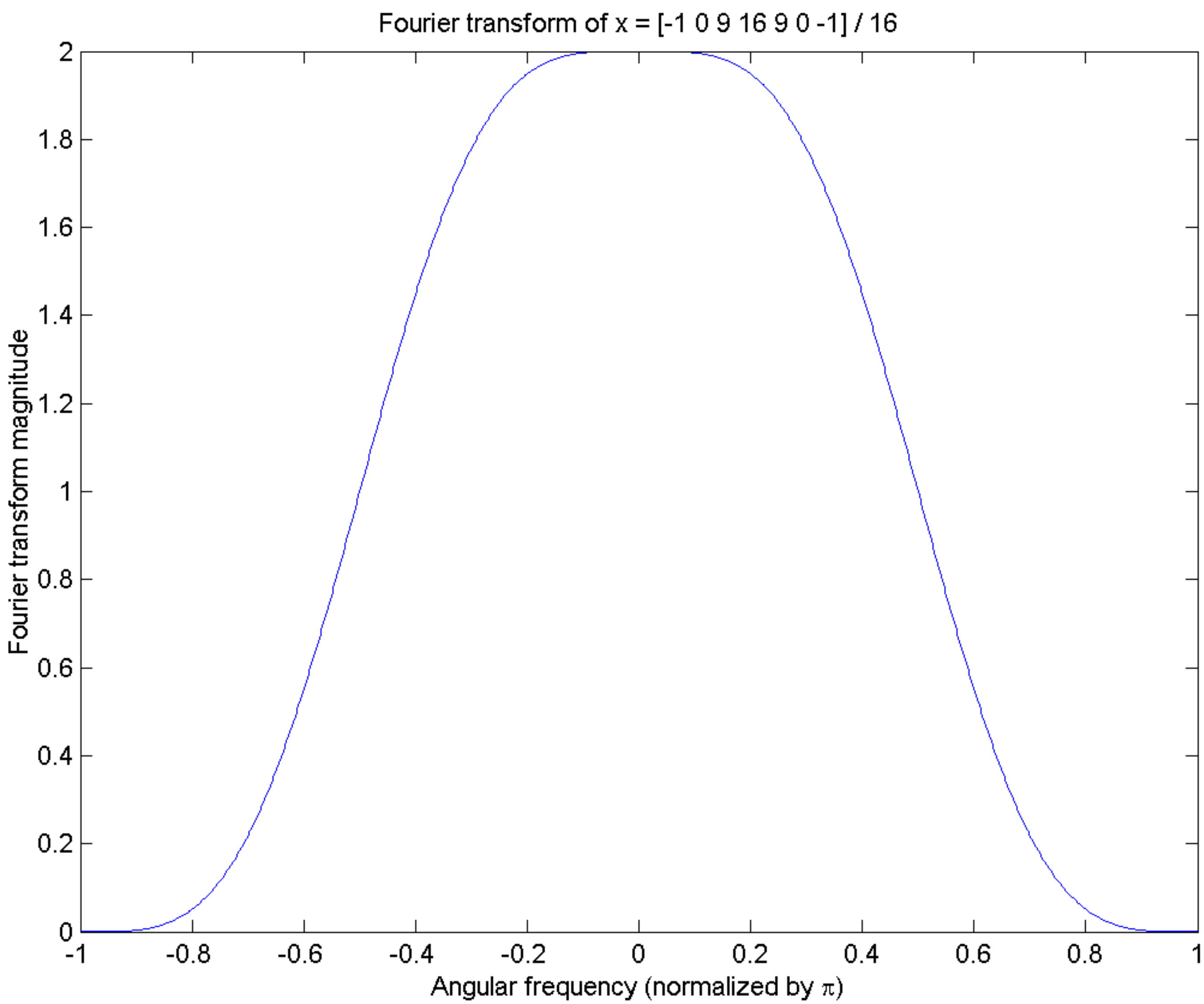


Upsampling

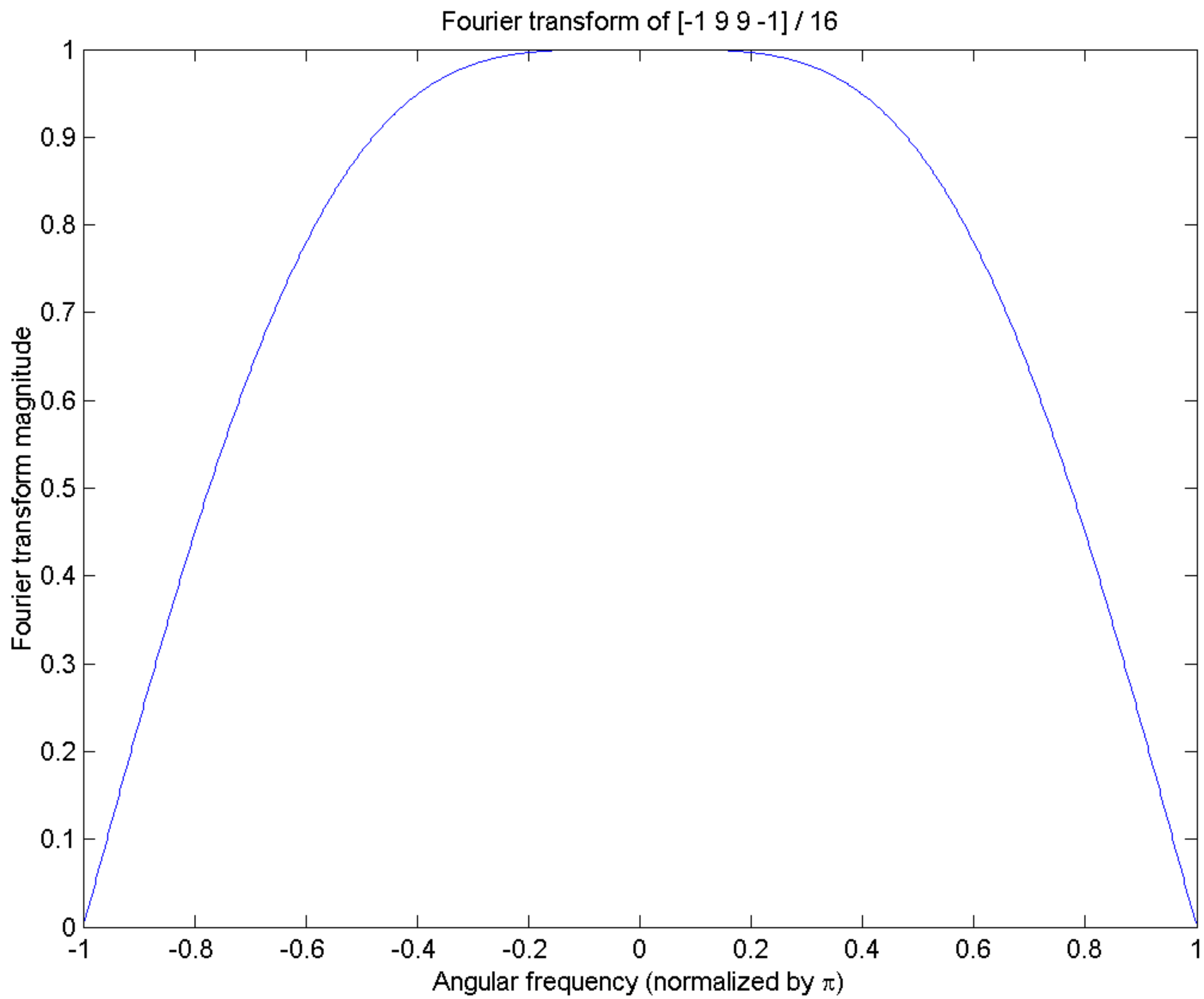


Fourier transform of [1/2 0 1/2 0]
Fourier transform magnitude
Angular frequency (normalized by π)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

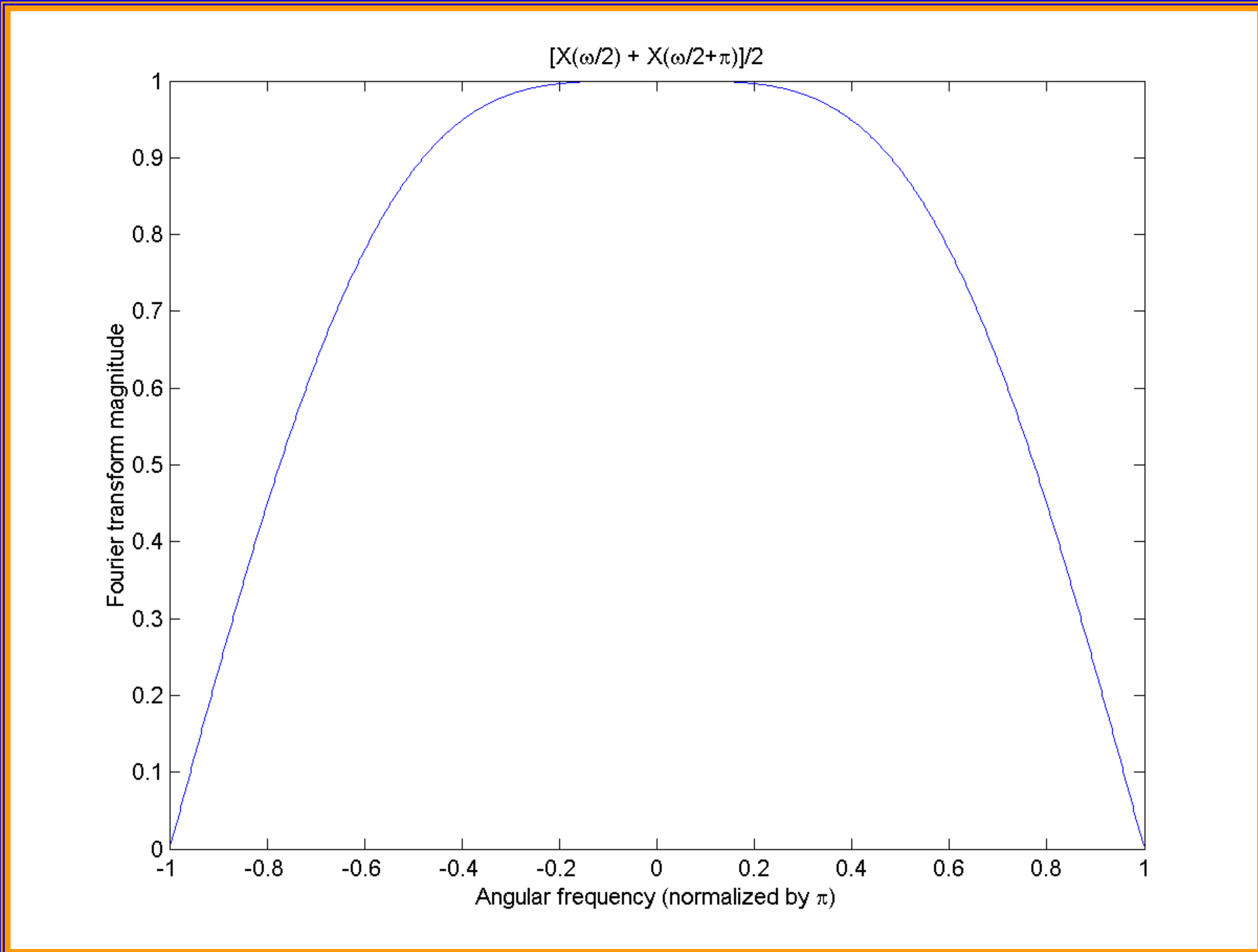
Downsampling



Downsampling



Downsampling

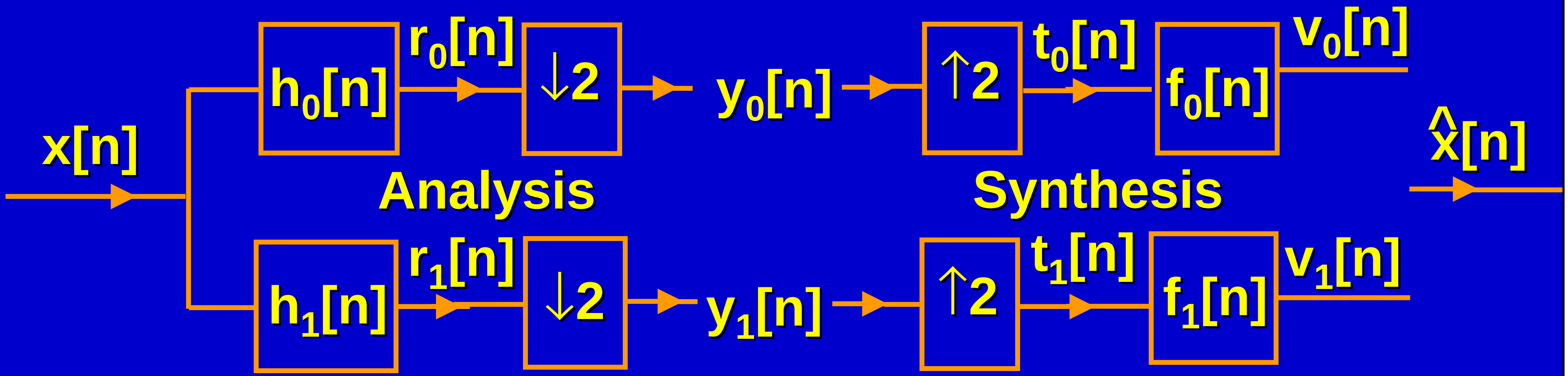


Course 18.327 and 1.130
Wavelets and Filter Banks

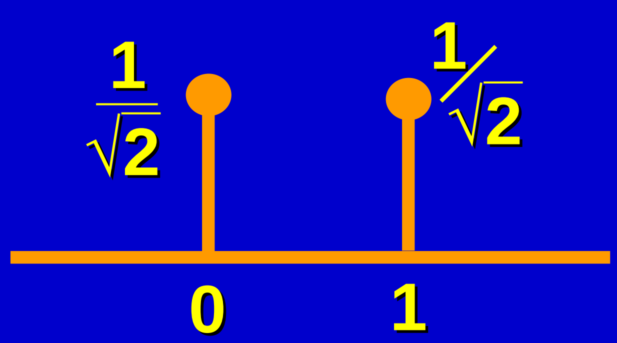
Filter Banks: time domain (Haar example) and frequency domain; conditions for alias cancellation and no distortion

Haar Filter Bank

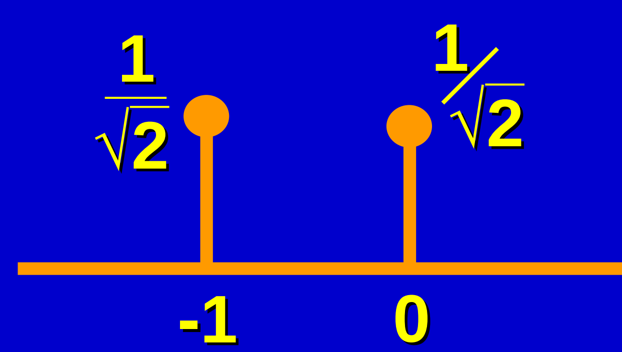
Simplest (non-trivial) example of a two channel FIR perfect reconstruction filter bank.



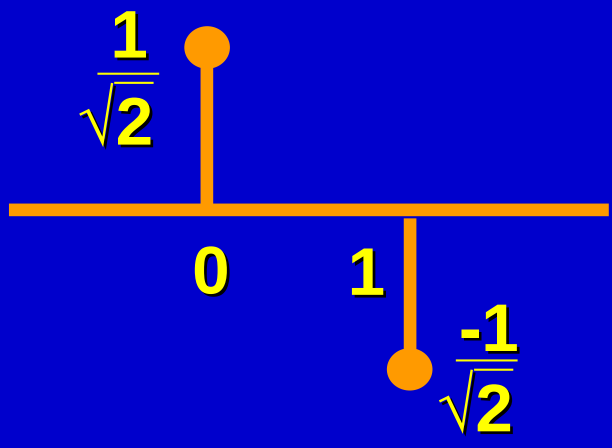
$h_0[n] =$



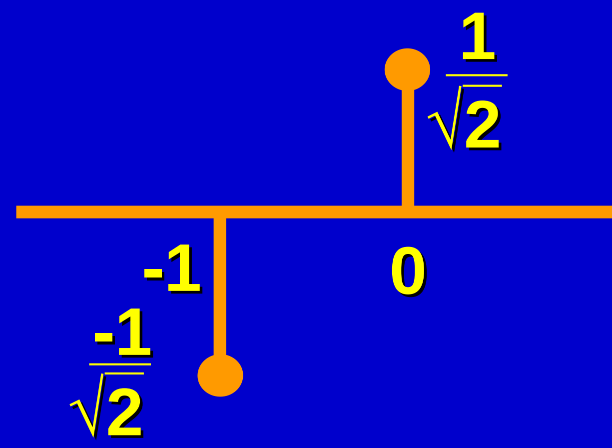
$f_0[n] =$



$h_1[n]$ =



$f_1[n]$ =



Analysis:

$$r_0[n] = \frac{1}{\sqrt{2}}\,(x[n] + x[n-1])$$ **lowpass filter**

$$y_0[n] = r_0[2n]$$ **downsampler**

$$y_0[n] = \frac{1}{\sqrt{2}}\,(x[2n] + x[2n-1])$$ ------------------- j

Similarly

$$y_1[n] = \frac{1}{\sqrt{2}}\,(x[2n] - x[2n-1])$$ ------------------- k

Matrix form

$$
\begin{bmatrix} \vdots \\ y_0[0] \\ y_0[1] \\ \vdots \\ \hline \vdots \\ y_1[0] \\ y_1[1] \\ \vdots \end{bmatrix}
= \frac{1}{\sqrt{2}}
\begin{bmatrix}
 & & & \vdots & & & \\
\cdots & 1 & 1 & & 0 & 0 & \cdots \\
\cdots & 0 & 0 & & 1 & 1 & \cdots \\
 & & & \vdots & & & \\ \hline
 & & & \vdots & & & \\
\cdots & -1 & 1 & & 0 & 0 & \cdots \\
\cdots & 0 & 0 & & -1 & 1 & \cdots \\
 & & & \vdots & & &
\end{bmatrix}
\begin{bmatrix} \vdots \\ x[-1] \\ x[0] \\ x[1] \\ \\ x[2] \\ \vdots \\ \vdots \end{bmatrix}
$$

$$
\begin{bmatrix} y_o \\ \hline y_1 \end{bmatrix} = \begin{bmatrix} L \\ \hline B \end{bmatrix} x \qquad \text{-----------------} \quad \mathrm{I}
$$

Synthesis

$$t_0[n] = \begin{cases} y_0[n/2] & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad \text{upsampler}$$

$$v_0[n] = \tfrac{1}{\sqrt{2}} \left(t_0[n+1] + t_0[n] \right) \qquad \text{lowpass filter}$$

$$= \begin{cases} \frac{1}{\sqrt{2}} y_0[n/2] & n \text{ even} \\ \frac{1}{\sqrt{2}} y_0\left[\frac{n+1}{2}\right] & n \text{ odd} \end{cases}$$

Similarly

$$v_1[n] = \begin{cases} \frac{1}{\sqrt{2}}\, y_1[n/2] & n \text{ even} \\ \frac{1}{\sqrt{2}}\, y_1[\frac{n+1}{2}] & n \text{ odd} \end{cases}$$

So, the reconstructed signal is

$$\hat{x}[n] = v_0[n] + v_1[n]$$

$$= \begin{cases} \frac{1}{\sqrt{2}}\,(y_0[n/2] + y_1[n/2]) & n \text{ even} \\ \frac{1}{\sqrt{2}}\,(y_0[\frac{n+1}{2}] - y_1[\frac{n+1}{2}]) & n \text{ odd} \end{cases}$$

i.e.

$$\hat{x}[2n-1] = \tfrac{1}{\sqrt{2}}\left(y_0[n] - y_1[n]\right) = x[2n-1]$$

from j and k

$$\hat{x}[2n] = \frac{1}{\sqrt{2}}\left(y_0[n] + y_1[n]\right) = x[2n]$$

So $\hat{x}[n] = x[n] \Rightarrow$ Perfect reconstruction!

In general, we will make all filters causal, so we will have

$$\hat{x}[n] = x[n - n_0] \Rightarrow \text{PR with delay}$$

Matrix form

$$\begin{bmatrix} \vdots \\ \hat{x}[-1] \\ \hat{x}[0] \\ \\ \hat{x}[1] \\ \hat{x}[2] \\ \vdots \end{bmatrix} = \frac{1}{\sqrt{2}} \left[\begin{array}{ccc|ccc} & \vdots & \vdots & \vdots & \vdots & \\ & 1 & 0 & -1 & 0 & \\ & 1 & 0 & 1 & 1 & \\ \cdots & & & \cdots & & \cdots \\ & 0 & 1 & 0 & -1 & \\ & 0 & 1 & -1 & 1 & \\ & \vdots & \vdots & \vdots & \vdots & \end{array}\right] \begin{bmatrix} \vdots \\ y_0[0] \\ y_0[1] \\ \vdots \\ \hline \vdots \\ y_1[0] \\ y_1[1] \\ \vdots \end{bmatrix}$$

$$\hat{x} = \left[\begin{array}{c|c} L^T & B^T \end{array}\right] \begin{bmatrix} y_0 \\ \hline y_1 \end{bmatrix}$$

m

Perfect reconstruction means that the synthesis bank is the inverse of the analysis bank.

$$\hat{x} = x \Rightarrow \underbrace{\left[\begin{array}{c|c} L^T & B^T \end{array}\right]}_{W^{-1}} \underbrace{\left[\begin{array}{c} L \\ \hline B \end{array}\right]}_{W} = I$$

1 2 3   1 2 3

W (Wavelet transform matrix)

In the Haar example, we have the special case

$$W^{-1} = W^T \rightarrow \text{orthogonal matrix}$$

So we have an orthogonal filter bank, where

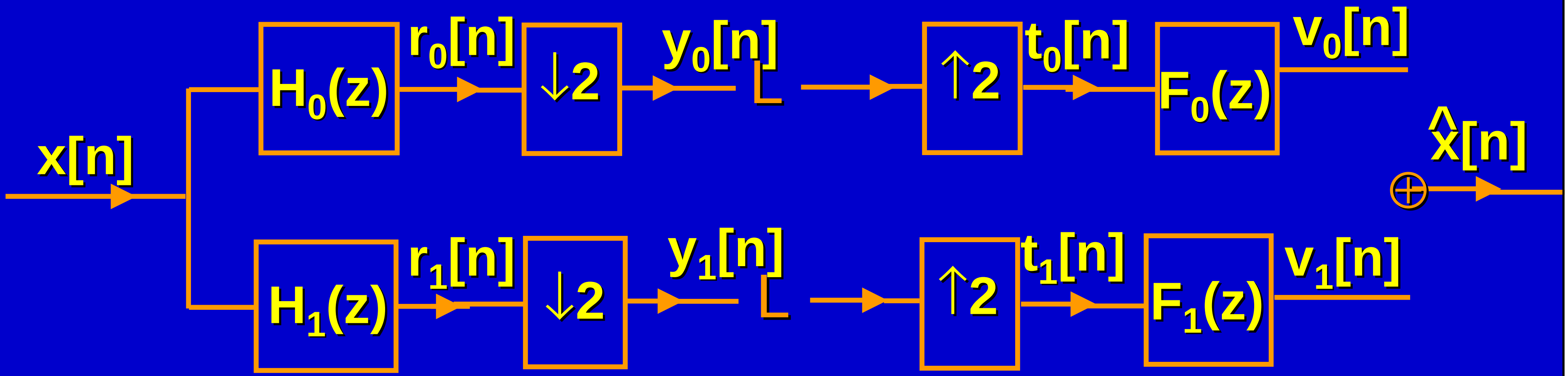
Synthesis bank = Transpose of Analysis bank

$f_0[n] = h_0[-n]$

$f_1[n] = h_1[-n]$

Perfect Reconstruction Filter Banks

General two-channel filter bank



z-transform definition:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

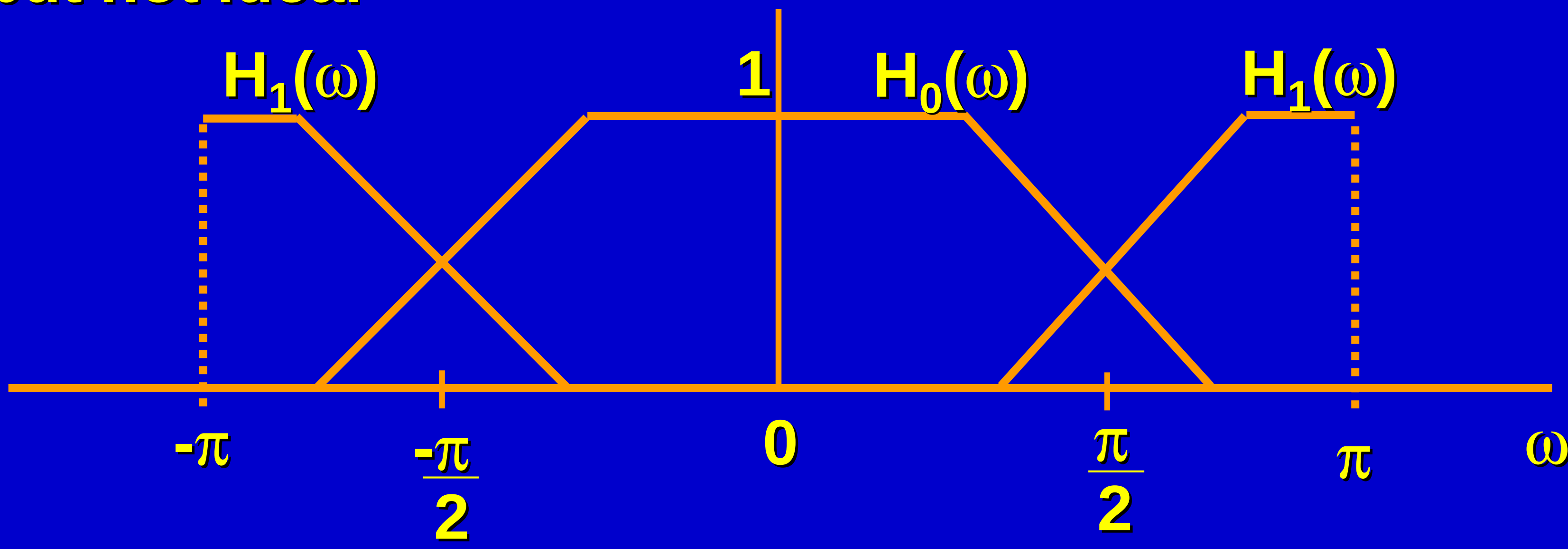
Put $z = e^{i\omega}$ to get DTFT

Perfect reconstruction requirement:

$$\hat{x}[n] = x[n - l] \quad (l \text{ time delays})$$

$$\hat{X}(z) = z^{-l} X(z)$$

$H_0(z)$ and $H_1(z)$ are normally lowpass and highpass, but not ideal



$\Rightarrow$ **Downsampling operation in each channel can produce aliasing**

Let's see why:

Lowpass channel has

$$Y_0(z) = \tfrac{1}{2}\{R_0(z^{1/2}) + R_0(-z^{1/2})\} \qquad \text{(downsampling)}$$

$$= \tfrac{1}{2}\{H_0(z^{1/2})\, X(z^{1/2}) + H_0(-z^{1/2})X(-z^{1/2})\}$$

In frequency domain:

$$X(z) \rightarrow X(\omega) \qquad \text{or} \quad X(e^{i\omega})$$

$$X(-z) \rightarrow X(\omega + \pi)$$

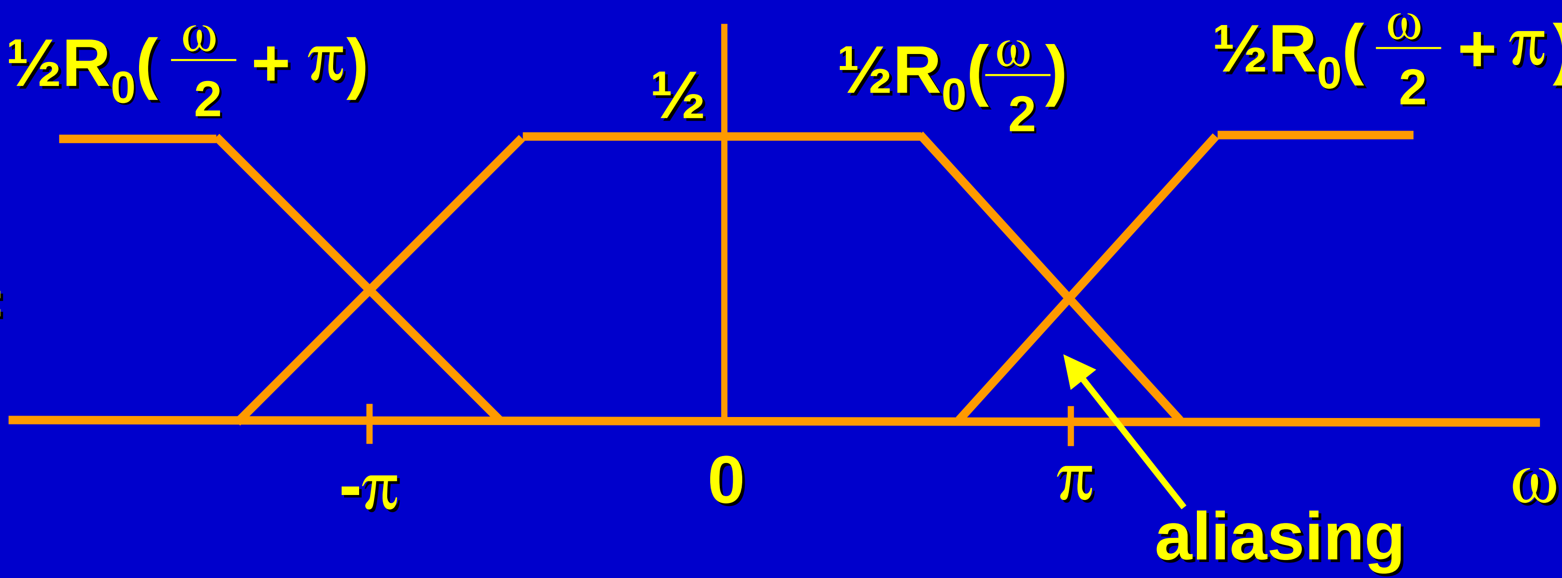
$$X(z^{1/2}) \rightarrow X\left(\frac{\omega}{2}\right)$$

$$Y_0(\omega) = \tfrac{1}{2}\left\{H_0\left(\frac{\omega}{2}\right) X\left(\frac{\omega}{2}\right) + H_0\left(\frac{\omega}{2} + \pi\right)X\left(\frac{\omega}{2} + \pi\right)\right\}$$

Suppose $X(\omega) = 1$ (input has all frequencies)

Then $R_0(\omega) = H_0(\omega)$, so that after downsampling we have

$Y_0(\omega) =$



Goal is to design $F_0(z)$ and $F_1(z)$ so that the overall system is just a simple delay - with no aliasing term:

$$V_0(z) + V_1(z) = z^{-l} X(z)$$

$$V_0(z) = F_0(z)\, T_0(z)$$

$$= F_0(z)\, Y_0(z^2) \qquad \text{(upsampling)}$$

$$= \tfrac{1}{2} F_0(z)\{ H_0(z)\, X(z) + H_0(-z)\, X(-z)\}$$

$$V_1(z) = \tfrac{1}{2} F_1(z)\{ H_1(z)\, X(z) + H_1(-z)\, X(-z)\}$$

So we want

$$\tfrac{1}{2} \{F_0(z)\, H_0(z) + F_1(z)\, H_1(z) \}\, X(z)$$

$$+ \qquad\qquad = z^{-l}\, X(z)$$

$$\tfrac{1}{2} \{F_0(z)\, H_0(-z) + F_1(z)\, H_1(-z) \}\, X(-z)$$

Compare terms in X(z) and X(-z):

1) **Condition for no distortion (terms in X (z) amount to a delay)**

$$F_0(z)\,H_0(z) + F_1(z)\,H_1(z) = 2z^{-l} \qquad \text{--------------- j}$$

2) **Condition for alias cancellation (no term in X(-z))**

$$F_0(z)\,H_0(-z) + F_1(z)\,H_1(-z) = 0 \qquad \text{--------------- k}$$

To satisfy alias cancellation condition, choose

$$\begin{aligned} F_0(z) &= H_1(-z) \\ F_1(z) &= -H_0(-z) \end{aligned} \qquad \text{---------------------- l}$$

What happens in the time domain?

$$F_0(z) = H_1(-z) \qquad\qquad F_0(\omega) = H_1(\omega + \pi)$$

$$= \sum_n h_1[n]\,(-z)^{-n}$$

$$= \sum_n (-1)^n\, h_1[n]\, z^{-n}$$

So the filter coefficients are

$$f_0[n] = (-1)^n \quad h_1[n]$$

$$f_1[n] = (-1)^{n+1}\, h_0[n]$$

alternating signs rule

Example

$h_0[n] = \{ a_0, a_1, a_2\}$ $\qquad$ $f_0[n] = \{ b_0, -b_1, b_2 \}$

$h_1[n] = \{ b_0, b_1, b_2\}$ $\qquad$ $f_1[n] = \{-a_0, a_1, -a_2\}$

Product Filter

Define

$$P_0(z) = F_0(z)\,H_0(z) \quad \text{-------------------------------} \text{m}$$

Substitute $F_1(z) = -H_0(-z)$, $H_1(z) = F_0(-z)$

in the zero distortion condition (Equation j)

$$F_0(z)\,H_0(z) - F_0(-z)\,H_0(-z) = 2z^{-l}$$

i.e. $$P_0(z) - P_0(-z) = 2z^{-l} \quad \text{-----------------------------} \text{n}$$

Note: l **must be odd since LHS is an odd function.**

Normalized Product Filter

Define

$$P(z) = z^{l} P_0(z) \quad \text{-----------------------------} \; \text{o}$$

$$P(-z) = -z^{l} P_0(-z) \quad \text{since } l \text{ is odd}$$

So we can rewrite Equation n **as**

$$z^{-l} P(z) + z^{-l} P(-z) = 2z^{-l}$$

i.e.

$$P(z) + P(-z) = 2 \quad \text{---------------------------} \; \text{p}$$

This is the condition on the normalized product filter for Perfect Reconstruction.

Design Process

1. Design P(z) to satisfy Equation p. This gives $P_0(z)$. Note: P(z) is designed to be lowpass.
2. Factor $P_0(z)$ into $F_0(z)\,H_0(z)$. Use Equations l to find $H_1(z)$ and $F_1(z)$.

Note: Equation p requires all even powers of z (except z^0) to be zero:

$$\sum_n p[n]z^{-n} + \sum_n p[n](-z)^{-n} = 2$$

$$\Rightarrow\; p[n] = \begin{cases} 1 & ; \quad n = 0 \\ 0 & ; \quad \text{all even } n \ (n \neq 0) \end{cases}$$

For odd n, p[n] and –p[n] cancel.

The odd coefficients, p[n], are free to be designed according to additional criteria.

Example: Haar filter bank

$$H_0(z) = \frac{1}{\sqrt{2}}(1 + z^{-1}) \qquad H_1(z) = \frac{1}{\sqrt{2}}(1 - z^{-1})$$

$$F_0(z) = H_1(-z) = \frac{1}{\sqrt{2}}(1 + z^{-1})$$

$$F_1(z) = -H_0(-z) = \frac{-1}{\sqrt{2}}(1 - z^{-1})$$

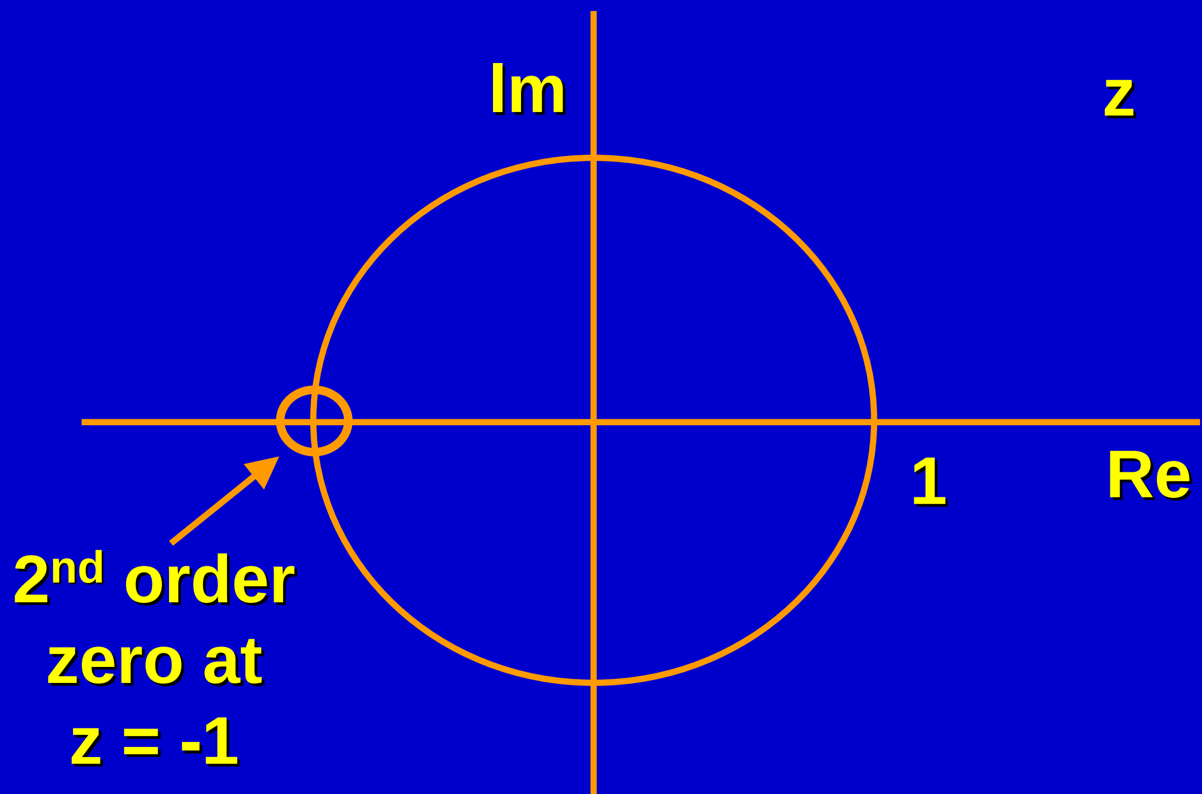
$$P_0(z) = F_0(z)\, H_0(z) = \frac{1}{2}(1 + z^{-1})^2$$

So the Perfect Reconstruction requirement is

$$P_0(z) - P_0(-z) = \frac{1}{2}(1 + 2z^{-1} + z^{-2}) - \frac{1}{2}(1 - 2z^{-1} + z^{-2})$$

$$= 2z^{-1} \quad \Rightarrow \quad l = 1$$

$$P(z) = z^{l}\, P_0(z) = \frac{1}{2}(1 + z)(1 + z^{-1})$$



Zeros of P(z):

$1 + z = 0$

$1 + z^{-1} = 0$

Course 18.327 and 1.130
Wavelets and Filter Banks

Filter Banks (contd.): perfect reconstruction; halfband filters and possible factorizations.

Product Filter

Example: Product filter of degree 6

$$P_0(z) = \frac{1}{16}(-1 + 9z^{-2} + 16z^{-3} + 9z^{-4} - z^{-6})$$

$$P_0(z) - P_0(-z) = 2z^{-3}$$

$\Rightarrow$ Expect perfect reconstruction with a 3 sample delay

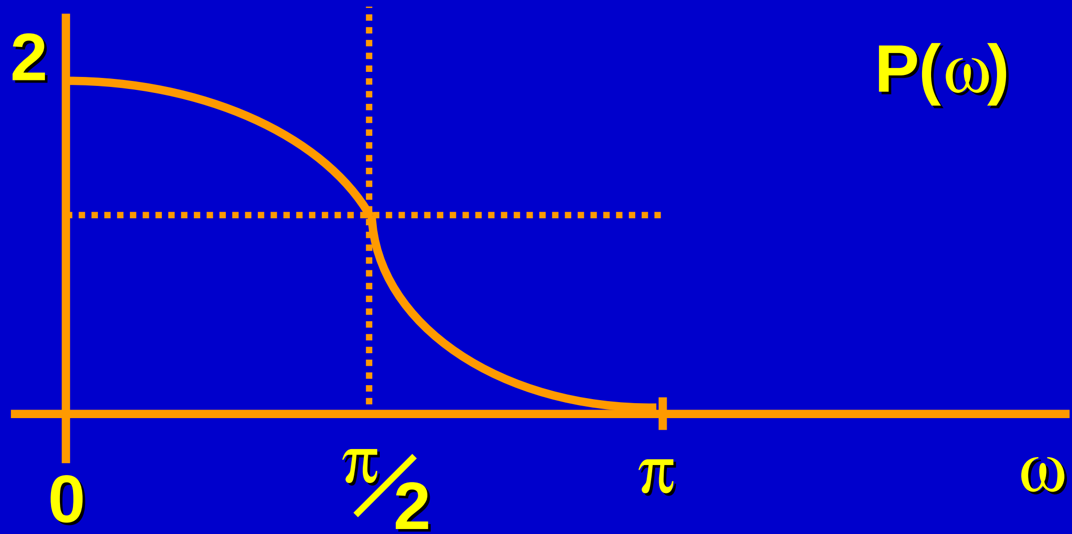
Centered form:

$$P(z) = z^3 P_0(z) = \frac{1}{16}(-z^3 + 9z + 16 + 9z^{-1} - z^{-3})$$

$P(z) + P(-z) = 2$ i.e. even part of $P(z)$ = const

In the frequency domain:

$$\boxed{P(\omega) + P(\omega + \pi) = 2}$$ Halfband Condition



Note antisymmetry about $\omega = \pi/2$

$P(\omega)$ is said to be a halfband filter.

How do we factor $P_0(z)$ into $H_0(z)\, F_0(z)$?

$$P_0(z) = 1/16(1 + z^{-1})^4(-1 + 4z^{-1} - z^{-2})$$

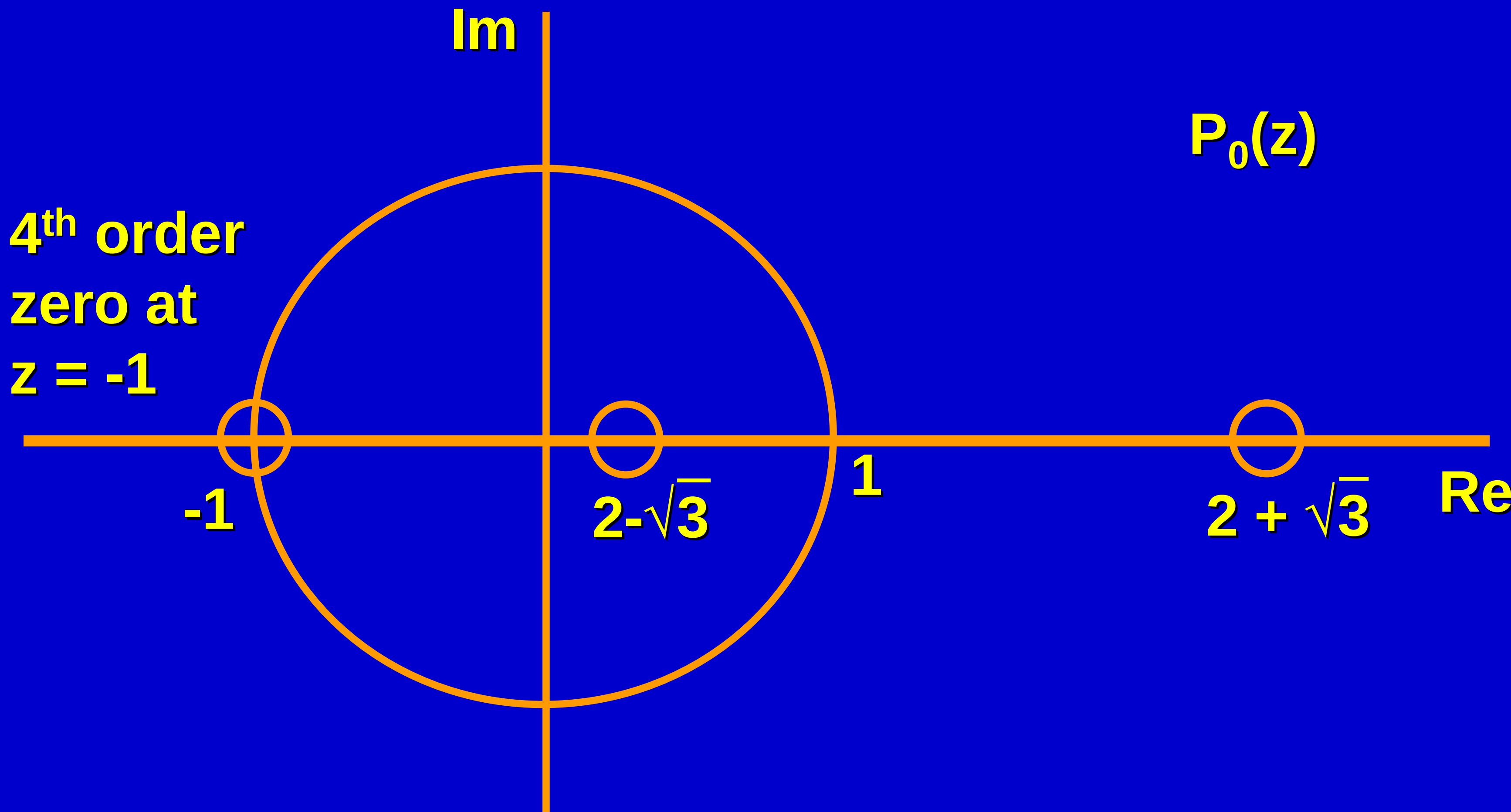
$$= -1/16(1 + z^{-1})^4(2 + \sqrt{3} - z^{-1})(2 - \sqrt{3} - z^{-1})$$

So $P_0(z)$ has zeros at

$z = -1$ (4^{th} order)

$z = 2 \pm \sqrt{3}$

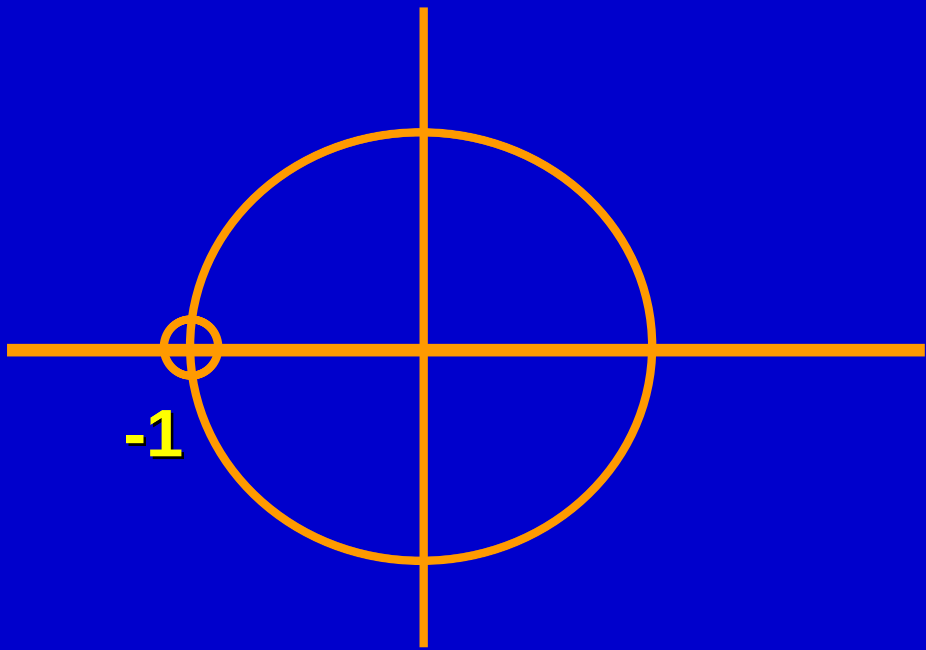
Note: $2 + \sqrt{3} = \dfrac{1}{2 - \sqrt{3}}$

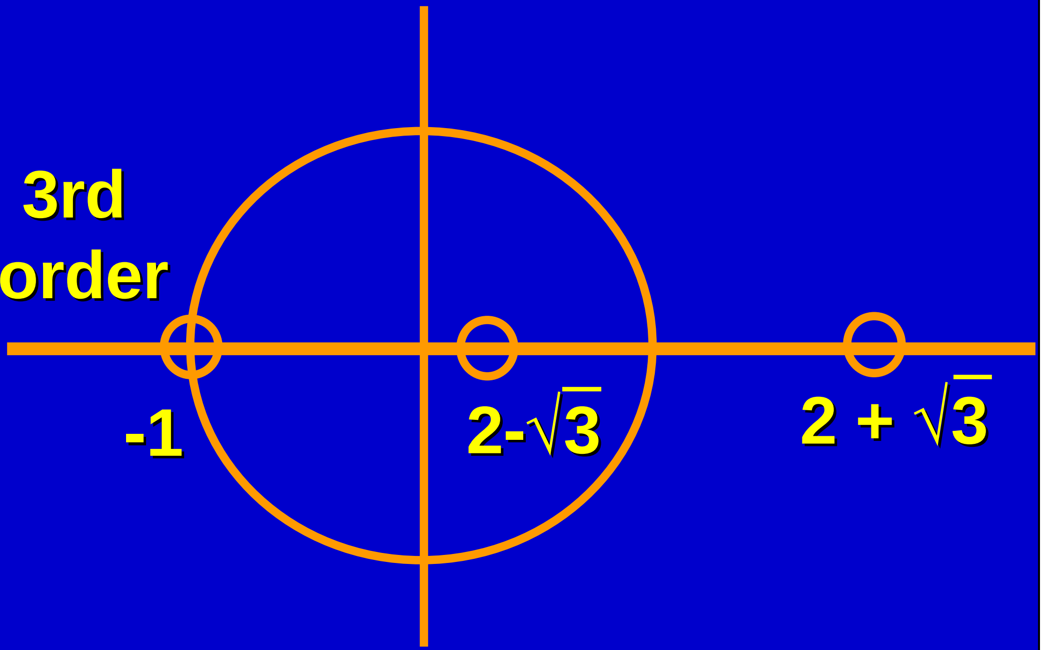


Some possible factorizations

	$H_0(z)$ (or $F_0(z)$)	$F_0(z)$ (or $H_0(z)$)
(a)	1	$-1/16(1 + z^{-1})^4(2 + \sqrt{3} - z^{-1})(2 - \sqrt{3} - z^{-1})$
(b)	$\frac{1}{2}(1 + z^{-1})$	$-1/8(1 + z^{-1})^3(2 + \sqrt{3} - z^{-1})(2 - \sqrt{3} - z^{-1})$
(c)	$\frac{1}{4}(1 + z^{-1})^2$	$-1/4(1 + z^{-1})^2(2 + \sqrt{3} - z^{-1})(2 - \sqrt{3} - z^{-1})$
(d)	$\frac{1}{2}(1 + z^{-1})(2 + \sqrt{3} - z^{-1})$	$-1/8(1 + z^{-1})^3(2 - \sqrt{3} - z^{-1})$
(e)	$1/8(1 + z^{-1})^3$	$-1/2(1 + z^{-1})(2 + \sqrt{3} - z^{-1})(2 - \sqrt{3} - z^{-1})$
(f)	$\frac{(\sqrt{3} - 1)}{4\sqrt{2}}(1 + z^{-1})^2(2 + \sqrt{3} - z^{-1})$	$\frac{-\sqrt{2}}{4(\sqrt{3} - 1)}(1 + z^{-1})^2(2 - \sqrt{3} - z^{-1})$
(g)	$1/16(1 + z^{-1})^4$	$-(2 + \sqrt{3} - z^{-1})(2 - \sqrt{3} - z^{-1})$

Case (b) -- Symmetric filters (linear phase)





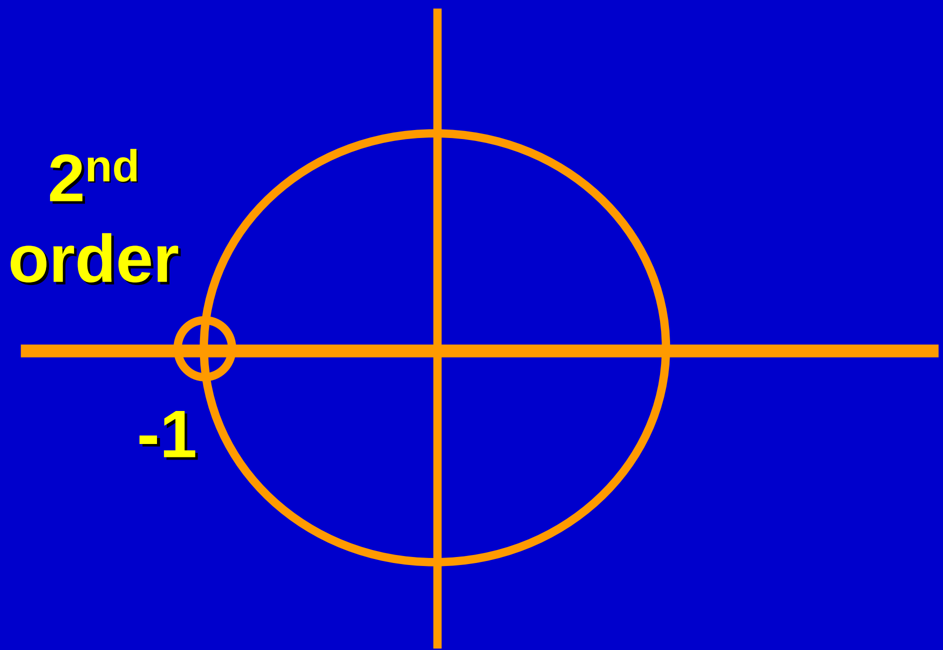
filter length = 2

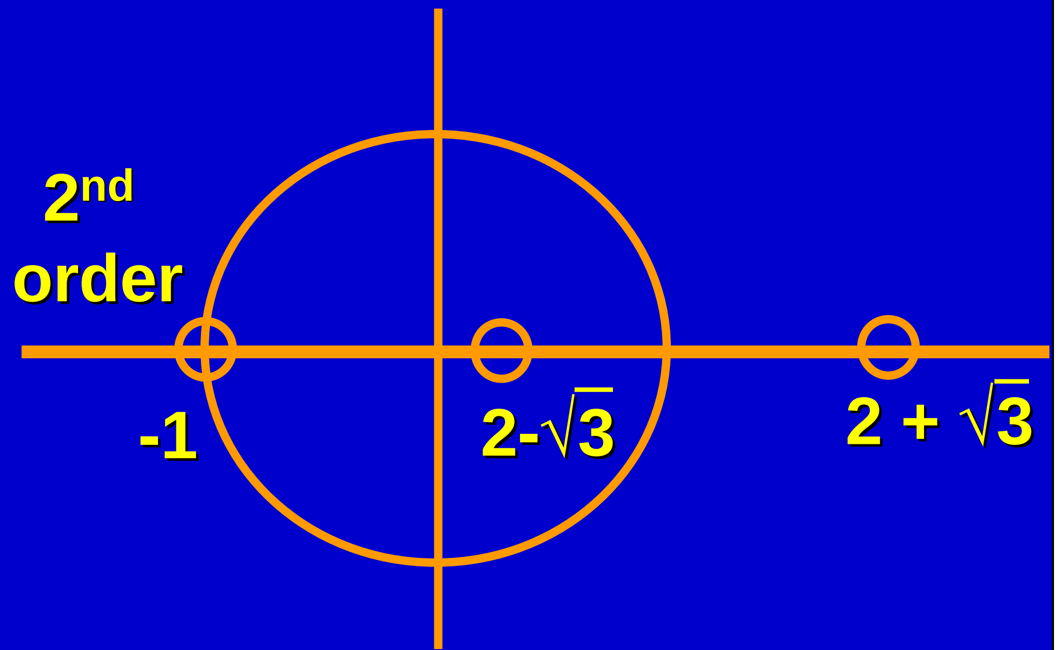
$\frac{1}{2}\{1, 1\}$

filter length = 6

$\frac{1}{8}\{-1, 1, 8, 8, 1, -1\}$

Case (c) -- Symmetric filters (linear phase)





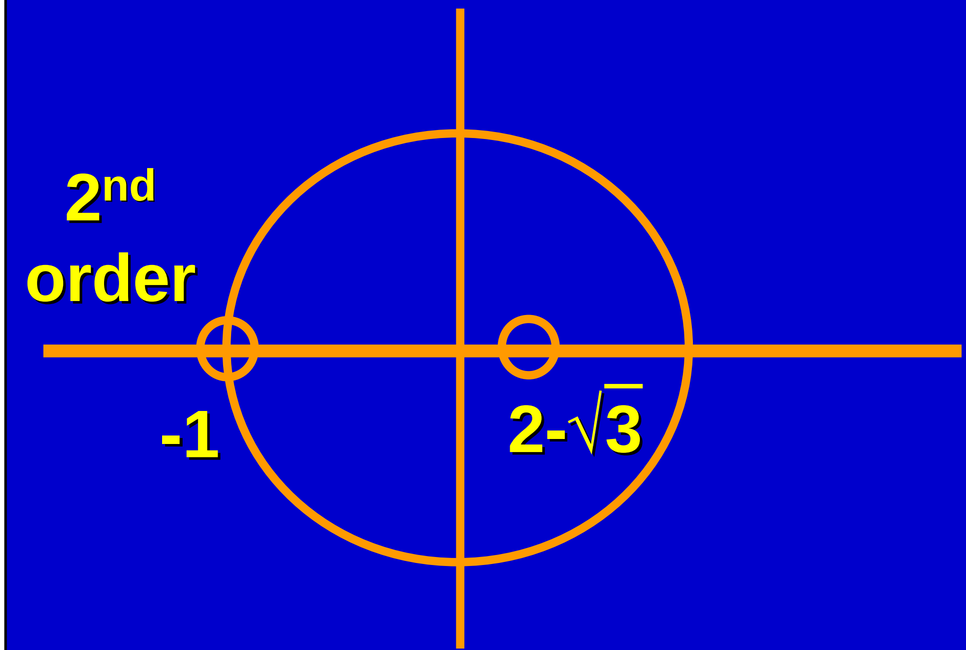
filter length = 3

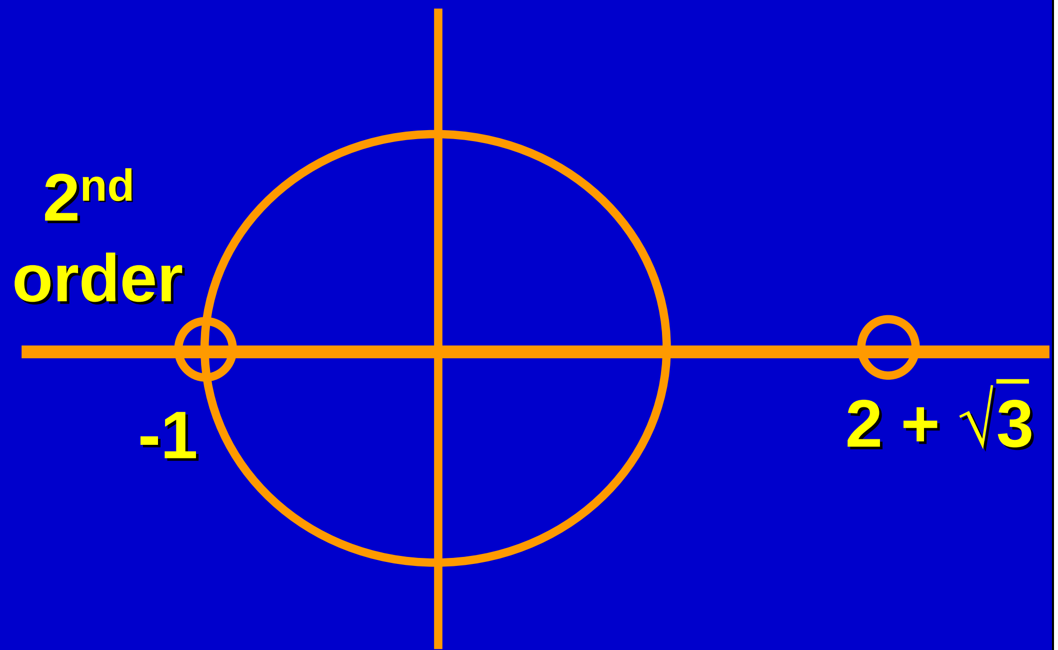
¼ { 1, 2, 1 }

filter length = 5

¼ { -1, 2, 6, 2, -1}

Case (f) -- Orthogonal filters (minimum phase/maximum phase)





filter length = 4

filter length = 4

$$\frac{1}{4\sqrt{2}}\left(1+\sqrt{3},\ 3+\sqrt{3},\ 3-\sqrt{3},\ 1-\sqrt{3}\right) \qquad \frac{1}{4\sqrt{2}}\left(1-\sqrt{3},\ 3-\sqrt{3},\ 3+\sqrt{3},\ 1+\sqrt{3}\right)$$

Note that, in this case, one filter is the flip (transpose) of the other: $f_0[n] = h_0[3 - n]$

$$F_0(z) = z^{-3} H_0(z^{-1})$$

General form of product filter (to be derived later):

$$P(z) = 2\left(\frac{1+z}{2}\right)^p \left(\frac{1+z^{-1}}{2}\right)^p \sum_{k=0}^{p-1} \binom{p+k-1}{k} \left(\frac{1-z}{2}\right)^k \left(\frac{1-z^{-1}}{2}\right)^k$$

$$P_0(z) = z^{-(2p-1)}\, P(z)$$

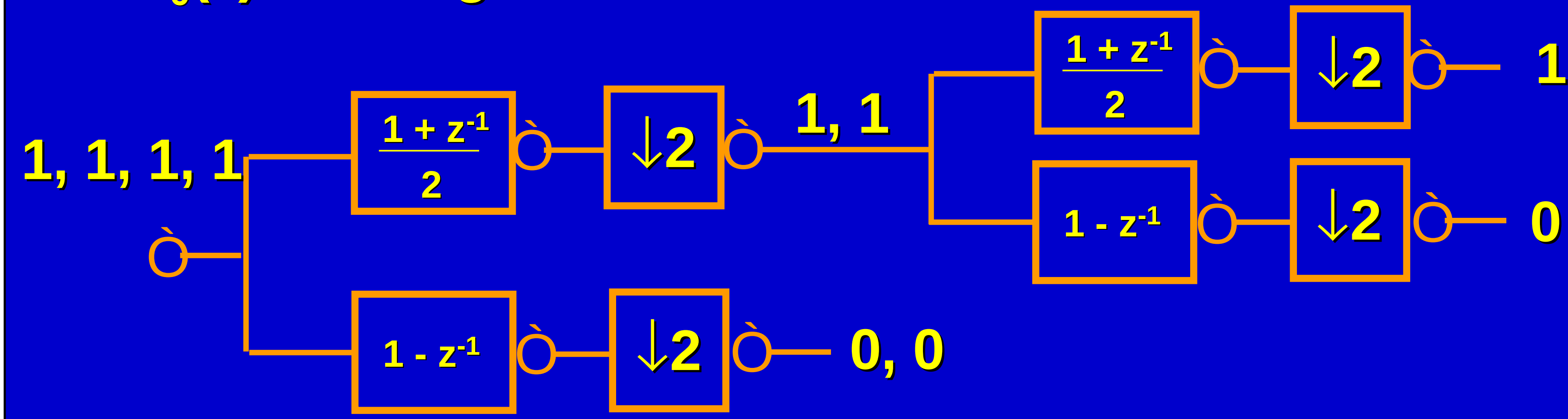
$$= \underbrace{(1+z^{-1})^{2p}}_{\substack{\text{Binomial}\\ \text{(spline)}\\ \text{filter}}} \underbrace{\frac{1}{2^{2p-1}} \sum_{k=0}^{p-1} \binom{p+k-1}{k} (-1)^k z^{-(p-1)+k} \left(\frac{1-z^{-1}}{2}\right)^{2k}}_{\substack{Q(z)\\ \text{Cancels all odd powers}\\ \text{except } z^{-(2p-1)}}}$$

$P_0(z)$ has 2p zeros at π (important for stability of iterated filter bank.)

Q(z) factor is needed to ensure perfect reconstruction.

$p = 1$

$P_0(z)$ has degree 2 $\rightarrow$ leads to Haar filter bank.



$F_0(z) = 1 + z^{-1}$, $H_0(z) = \frac{1 + z^{-1}}{2}$

Synthesis lowpass filter has 1 zero at π

$\rightarrow$ Leads to cancellation of constant signals in analysis highpass channel.

Additional zeros at π would lead to cancellation of higher order polynomials.

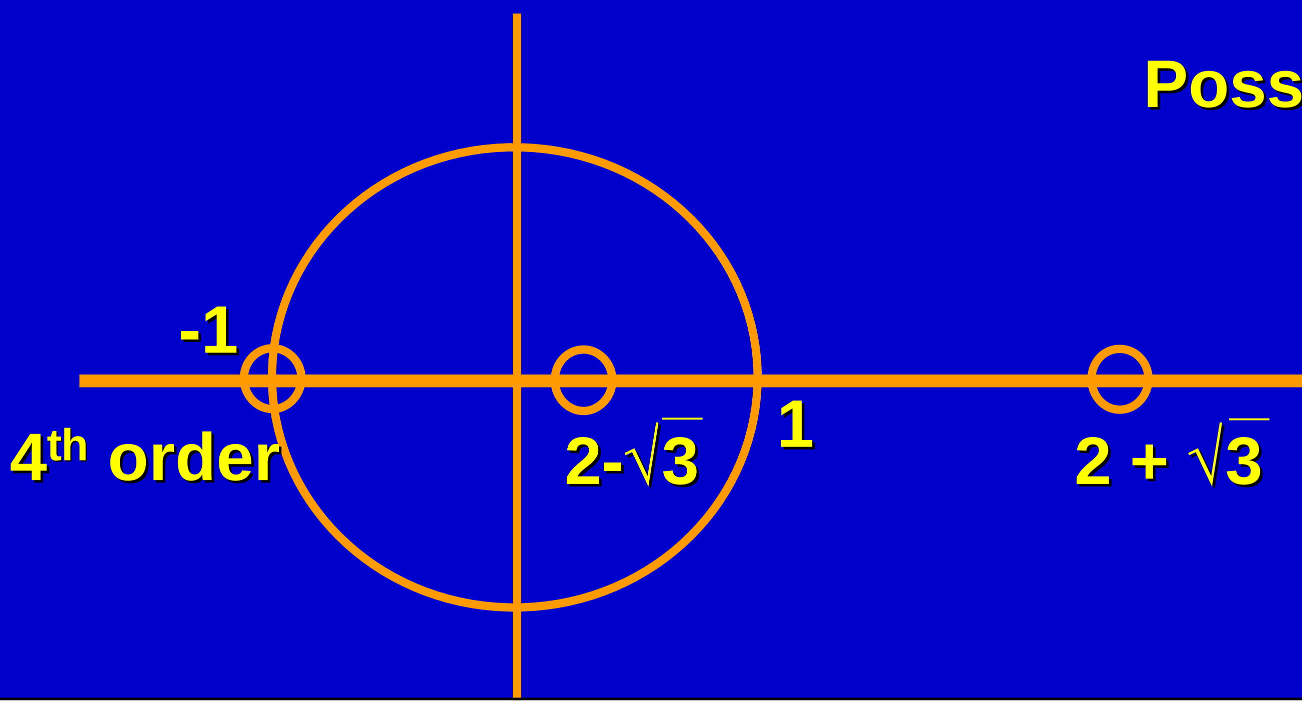
$p = 2$

$P_0(z)$ has degree $4p - 2 = 6$

$$P_0(z) = (1 + z^{-1})^4 \frac{1}{8} \left\{ \binom{1}{0} z^{-1} - \binom{2}{1} \left(\frac{1 - z^{-1}}{2}\right)^2 \right\}$$

$$= \frac{1}{16}(1 + z^{-1})^4(-1 + 4z^{-1} - z^{-2})$$

$$= \frac{1}{16}\{-1 + 9z^{-2} + 16z^{-3} + 9z^{-4} - z^{-6}\}$$



Possible factorizations

1/8 trivial

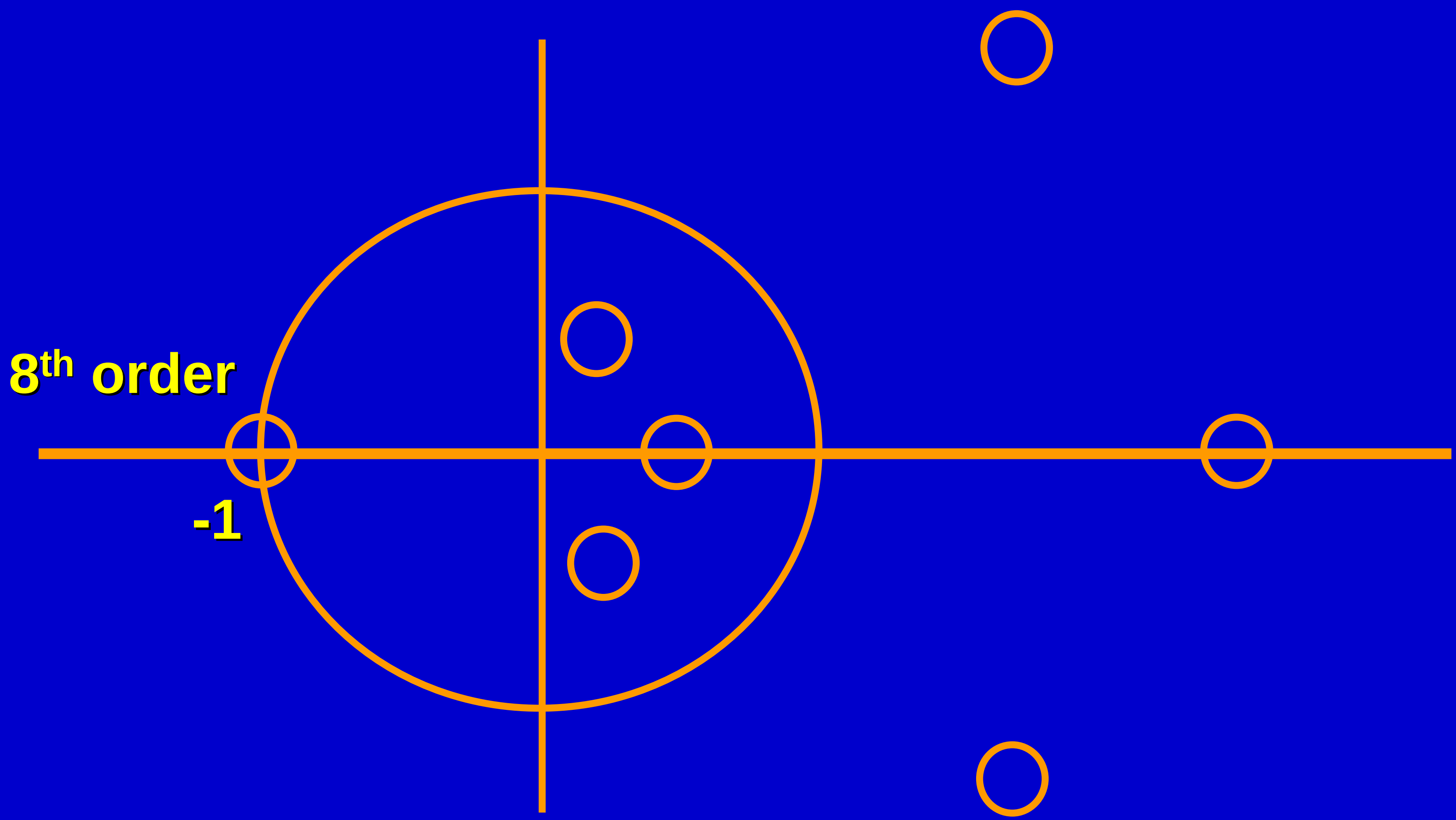
2/6 linear phase

3/5

4/4 orthogonal (Daubechies-4)

$p = 4$

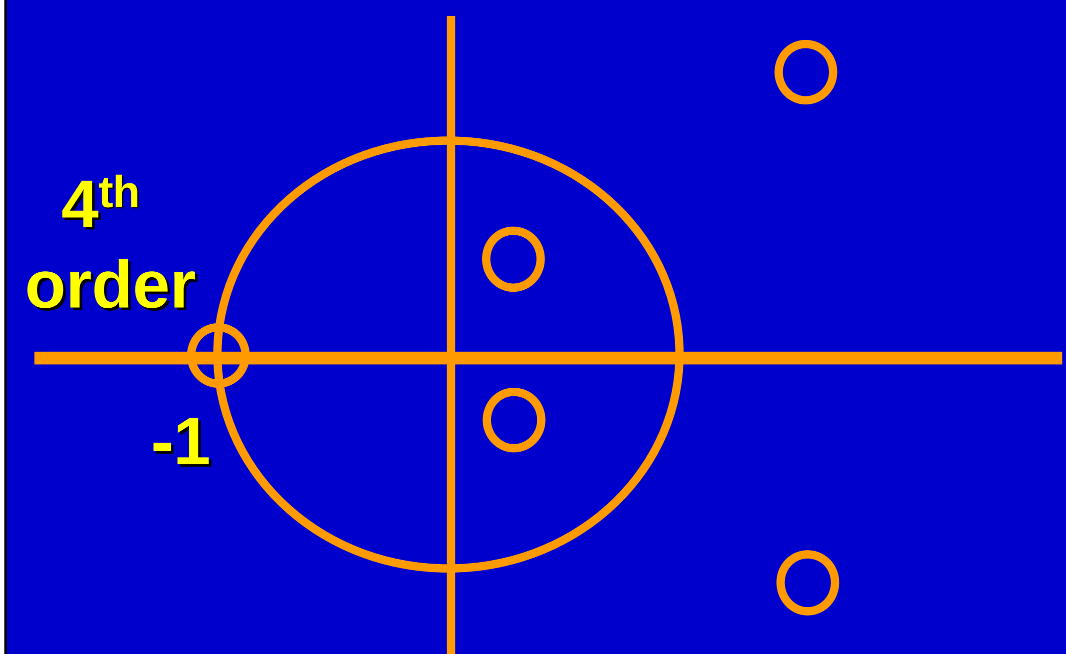
$P_0(z)$ has degree $4p - 2 = 14$

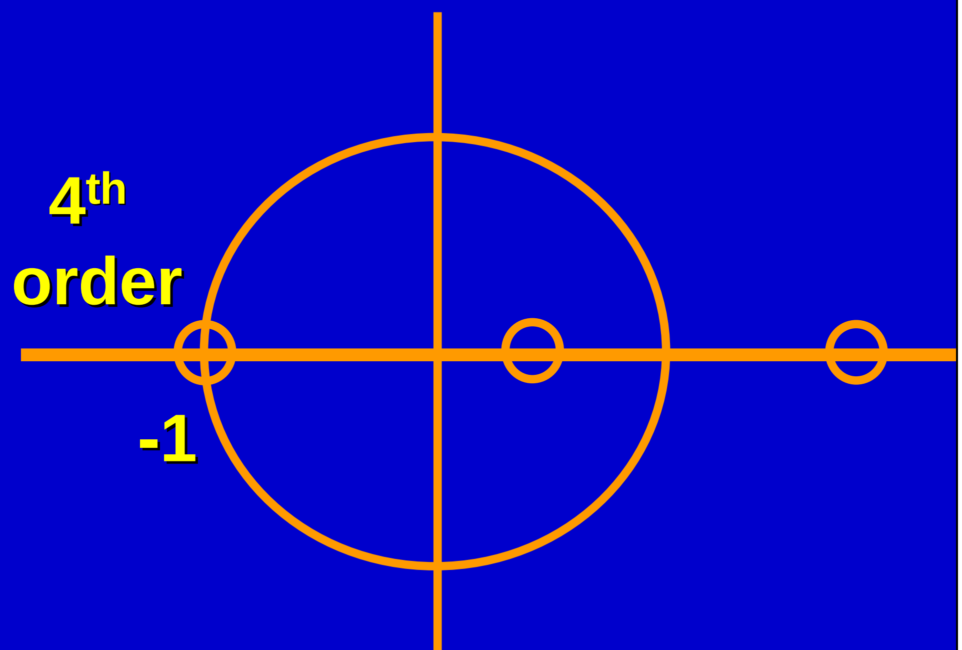


Common factorizations (p = 4):

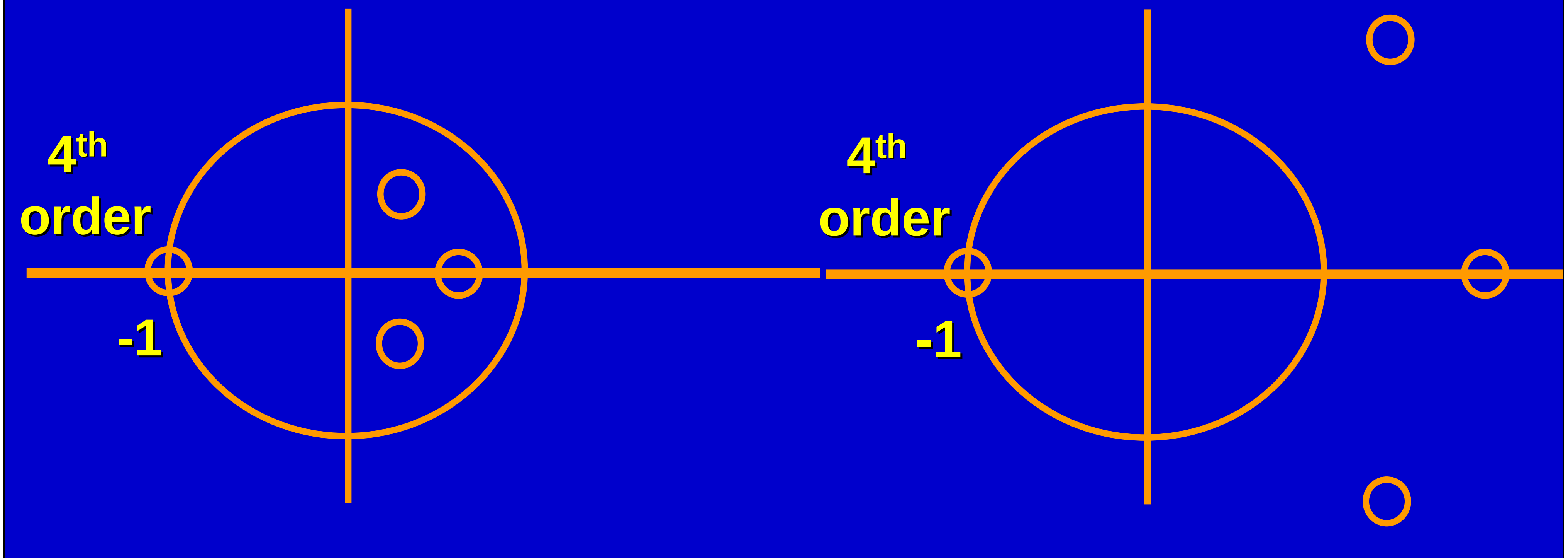
(a) 9/7

Known in Matlab as bior4.4





(b) 8/8 (Daubechies 8) -- Known in Matlab as db4

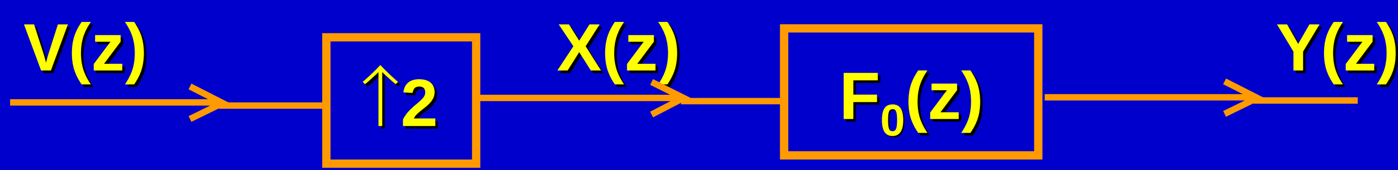


Why choose a particular factorization?

Consider the example with p = 2:

i. **One of the factors is halfband**

The trivial 1/8 factorization is generally not desirable, since each factor should have at least one zero at π. However, the fact that $F_0(z)$ is halfband is interesting in itself.



Let $F_0(z)$ be centered, for convenience. Then

$$F_0(z) = 1 + \text{odd powers of } z$$

Now

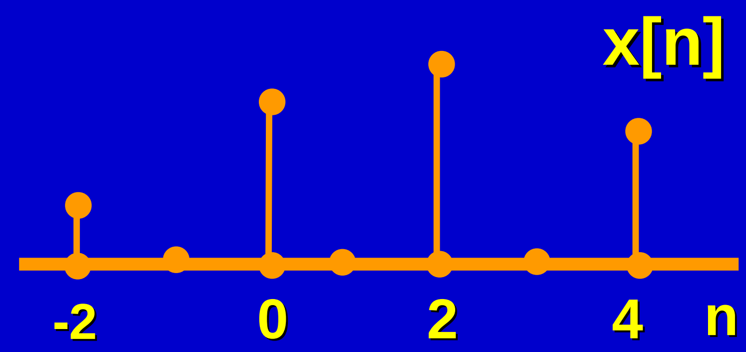
$$X(z) = V(z^2) = \text{even powers of } z \text{ only}$$

So

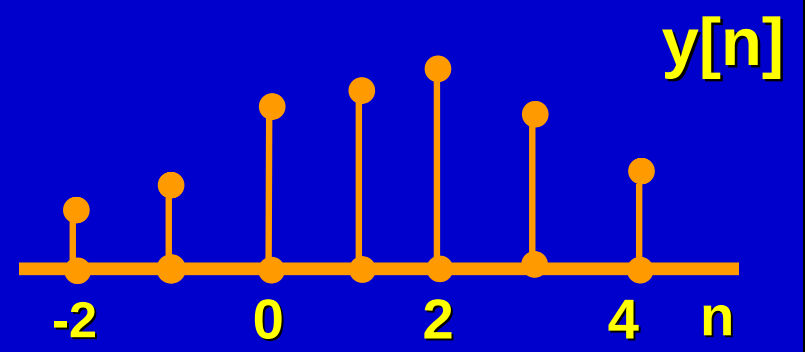
$$Y(z) = F_0(z)\, X(z)$$
$$= X(z) + \text{odd powers}$$

$$y[n] = \begin{cases} x[n] & ; \quad n \text{ even} \\ \sum\limits_{k \text{ odd}} f_0[k]\,x[n-k] & ; \quad n \text{ odd} \end{cases}$$

$\Rightarrow$ $f_0[n]$ is an interpolating filter



Another example: $f_0[n] = \dfrac{\sin\left(\frac{\pi}{2}\right)n}{\pi n}$
(ideal bandlimited interpolating filter)



ii. Linear phase factorization e.g. 2/6, 5/3

Symmetric (or antisymmetric) filters are desirable for many applications, such as image processing. All frequencies in the signal are delayed by the same amount i.e. there is no phase distortion.

$$h[n] \text{ linear phase} \Rightarrow A(\omega)e^{-i(\omega\alpha + \theta)}$$

$A(\omega)$: real

α: delays all frequencies by α samples

θ: 0 if symmetric; $\frac{\pi}{2}$ if antisymmetric

Linear phase may not necessarily be the best choice for audio applications due to preringing effects.

iii. Orthogonal factorization

This leads to a minimum phase filter and a maximum phase filter, which may be a better choice for applications such as audio. The orthogonal factorization leads to the Daubechies family of wavelets – a particularly neat and interesting case. 4/4 factorization:

$$H_0(z) = \frac{\sqrt{3} - 1}{4\sqrt{2}} (1 + z^{-1})^2[(2 + \sqrt{3}) - z^{-1}]$$

$$= \frac{1}{4\sqrt{2}} \{(1 + \sqrt{3}) + (3 + \sqrt{3})z^{-1} + (3 - \sqrt{3})z^{-2} + (1 - \sqrt{3})z^{-3}\}$$

$$F_0(z) = \frac{-\sqrt{2}}{4(\sqrt{3}-1)} (1 + z^{-1})^2[(2 - \sqrt{3}) - z^{-1}]$$

$$= \frac{\sqrt{3}-1}{4\sqrt{2}} z^{-3} (1 + z^2)[(2 + \sqrt{3}) - z]$$

$$= z^{-3} H_0 (z^{-1})$$

$$P(z) = z^{l} P_0(z)$$

$$= H_0(z)\, H_0(z^{-1})$$

From alias cancellation condition:

$$H_1(z) = F_0(-z) = -z^{-3}\, H_0(-z^{-1})$$

$$F_1(z) = -H_0(-z) = z^{-3}\, H_1(z^{-1})$$

Special Case: Orthogonal Filter Banks

Choose $H_1(z)$ so that

$$\boxed{H_1(z) = -z^{-N} H_0(-z^{-1})}$$

N odd

Time domain

$$\boxed{h_1[n] = (-1)^n h_0[N - n]}$$

$F_0(z) = H_1(-z) = z^{-N} H_0(z^{-1})$

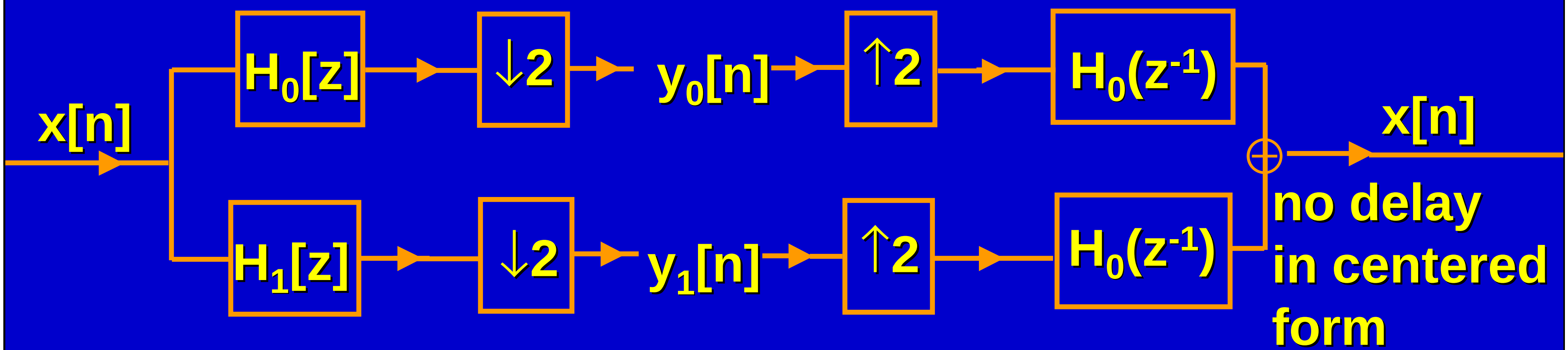
$\Rightarrow \quad f_0[n] = h_0[N - n]$

$F_1(z) = -H_0(-z) = z^{-N} H_1(z^{-1})$

$\Rightarrow f_1[n] = h_1[N - n]$

So the synthesis filters, $f_k[n]$, are just the time-reversed versions of the analysis filters, $h_k[n]$, with a delay.

Why is the Daubechies factorization orthogonal?
Consider the centered form of the filter bank:



Analysis bank
causal – only
negative powers
of z

Synthesis bank
anticausal – only
positive powers
of z

In matrix form:

Analysis

$$\begin{bmatrix} y_o \\ \hline y_1 \end{bmatrix} = \begin{bmatrix} L \\ \hline B \end{bmatrix} \begin{bmatrix} x \end{bmatrix}$$

1 2 3

W

Synthesis

$$\begin{bmatrix} x \end{bmatrix} = \left[\begin{array}{c|c} L^T & B^T \end{array}\right] \begin{bmatrix} y_0 \\ \hline y_1 \end{bmatrix}$$

1 2 3

W^T

So

$$x = W^T W x \text{ for any } x$$

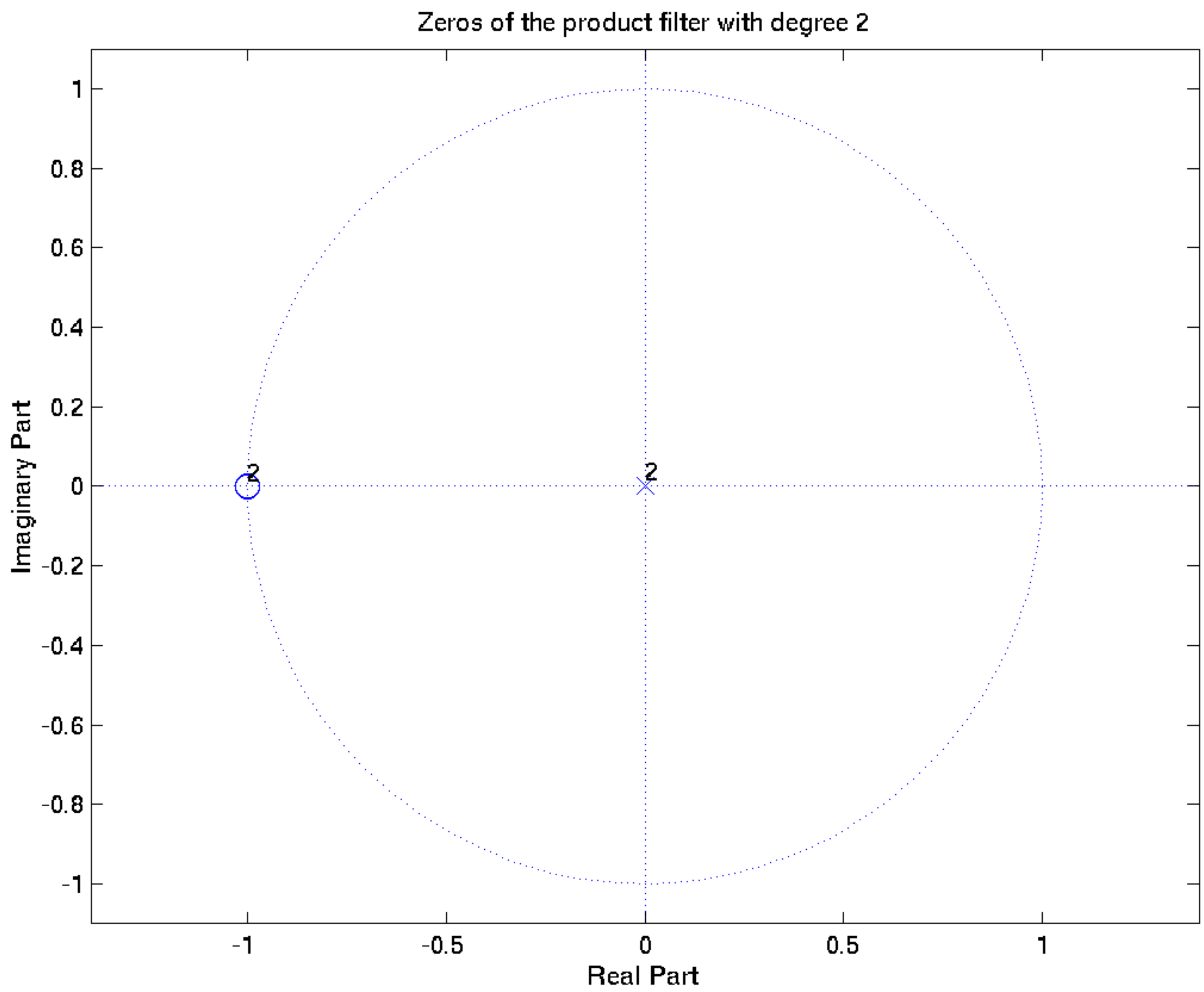
$$W^T W = I = W W^T$$

An important fact: symmetry prevents orthogonality

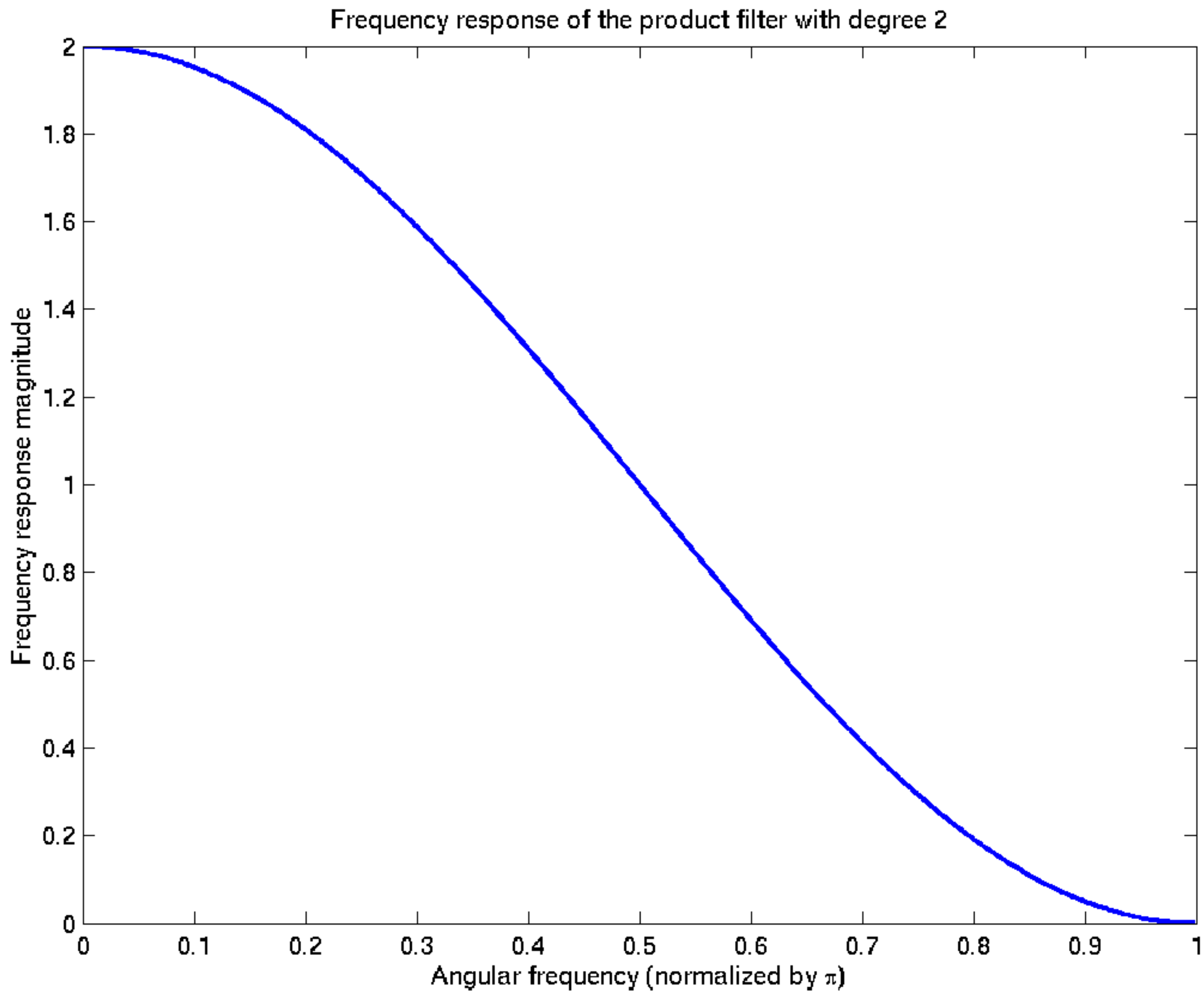
Matlab Example 2

1. Product filter examples

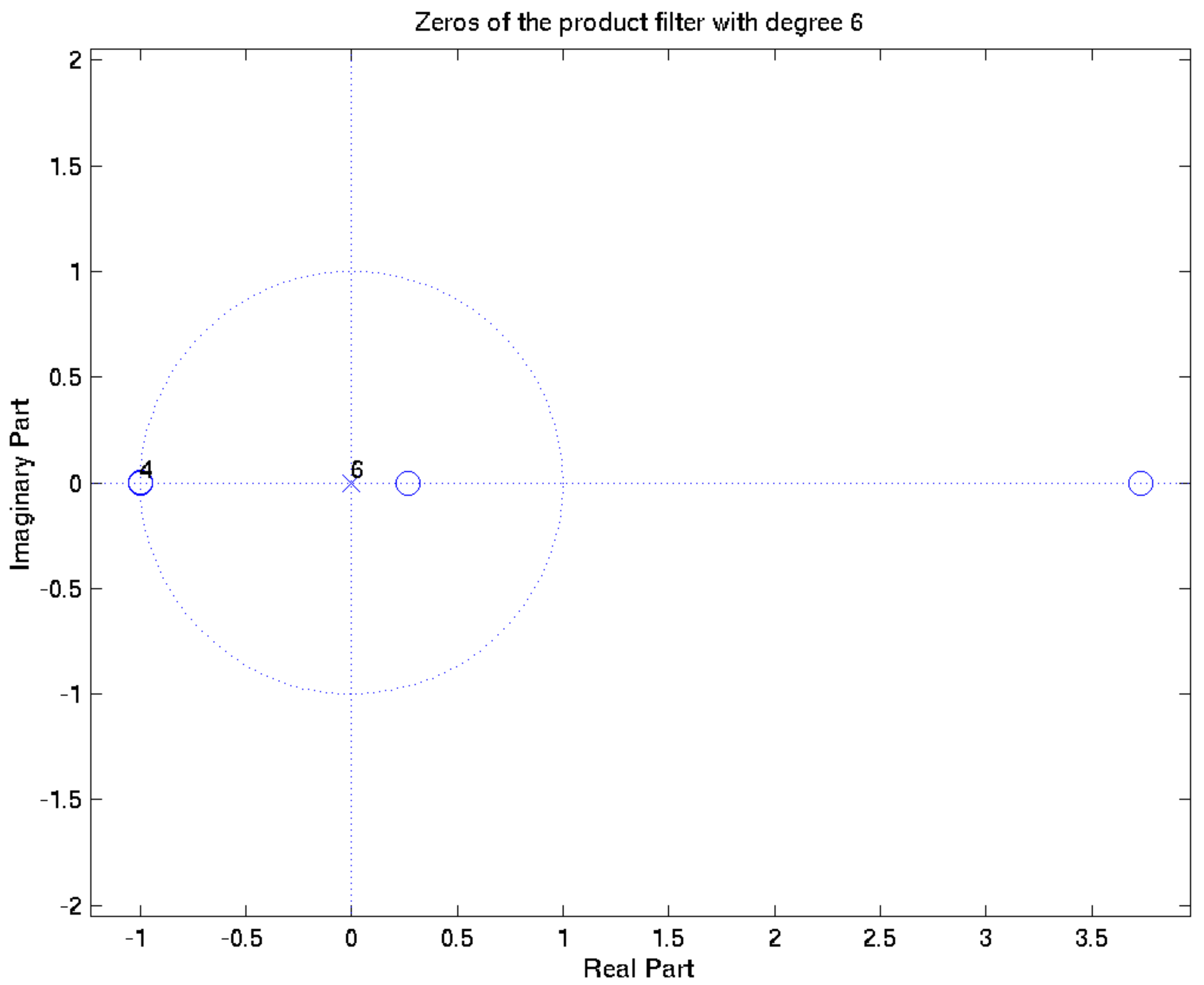
Degree-2 (p=1): pole-zero plot



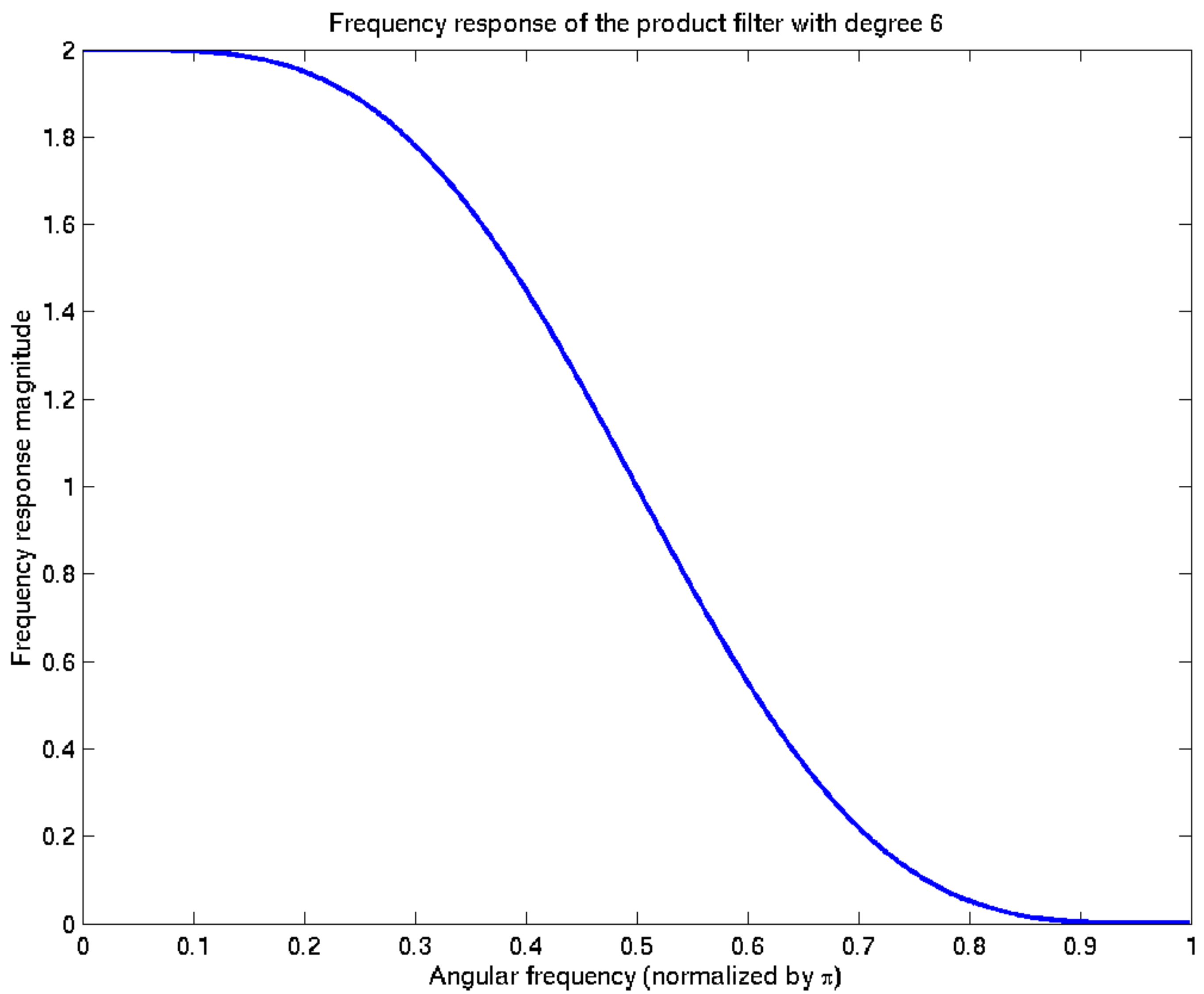
Degree-2 (p=1): Freq. response



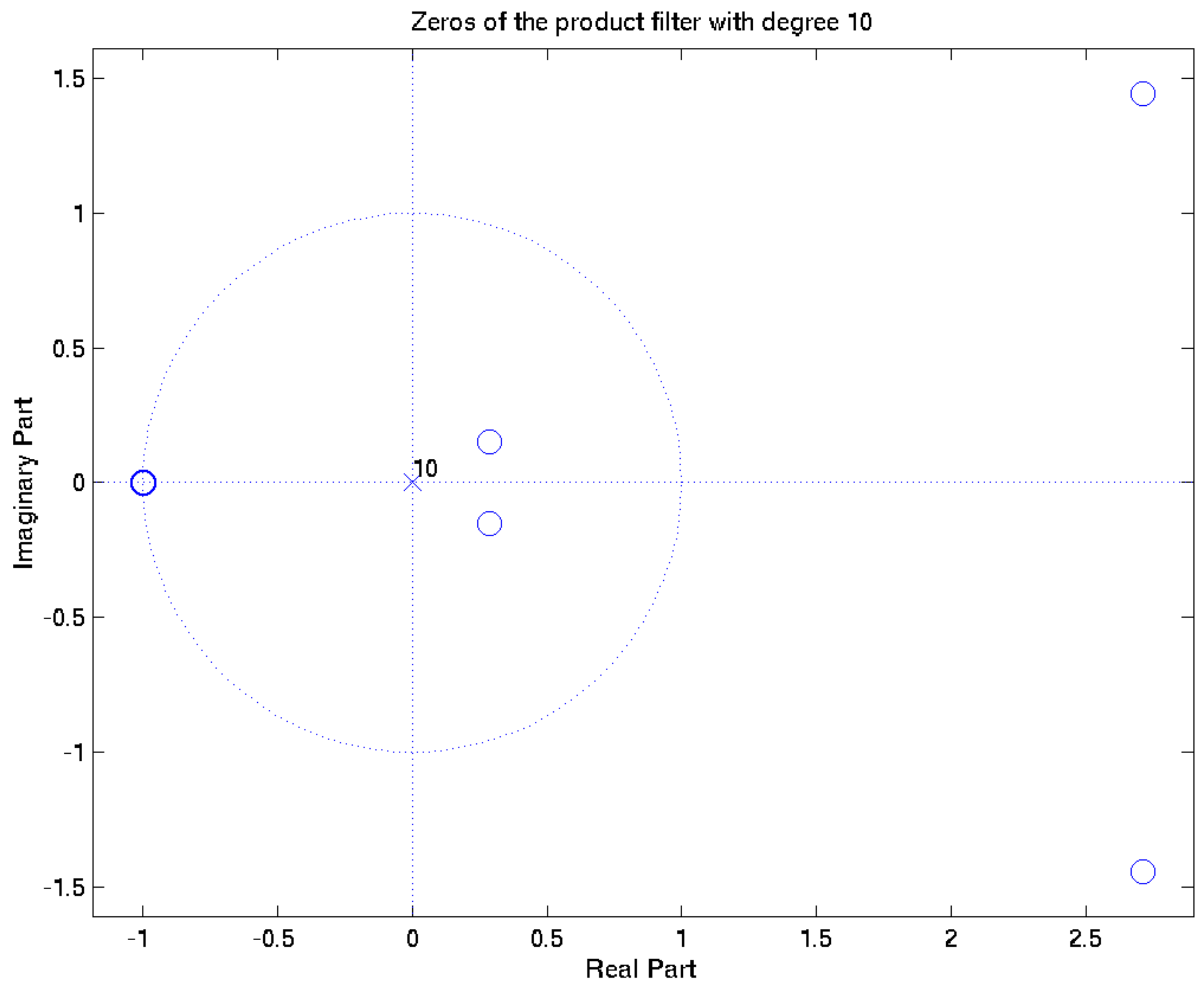
Degree-6 (p=2): pole-zero plot



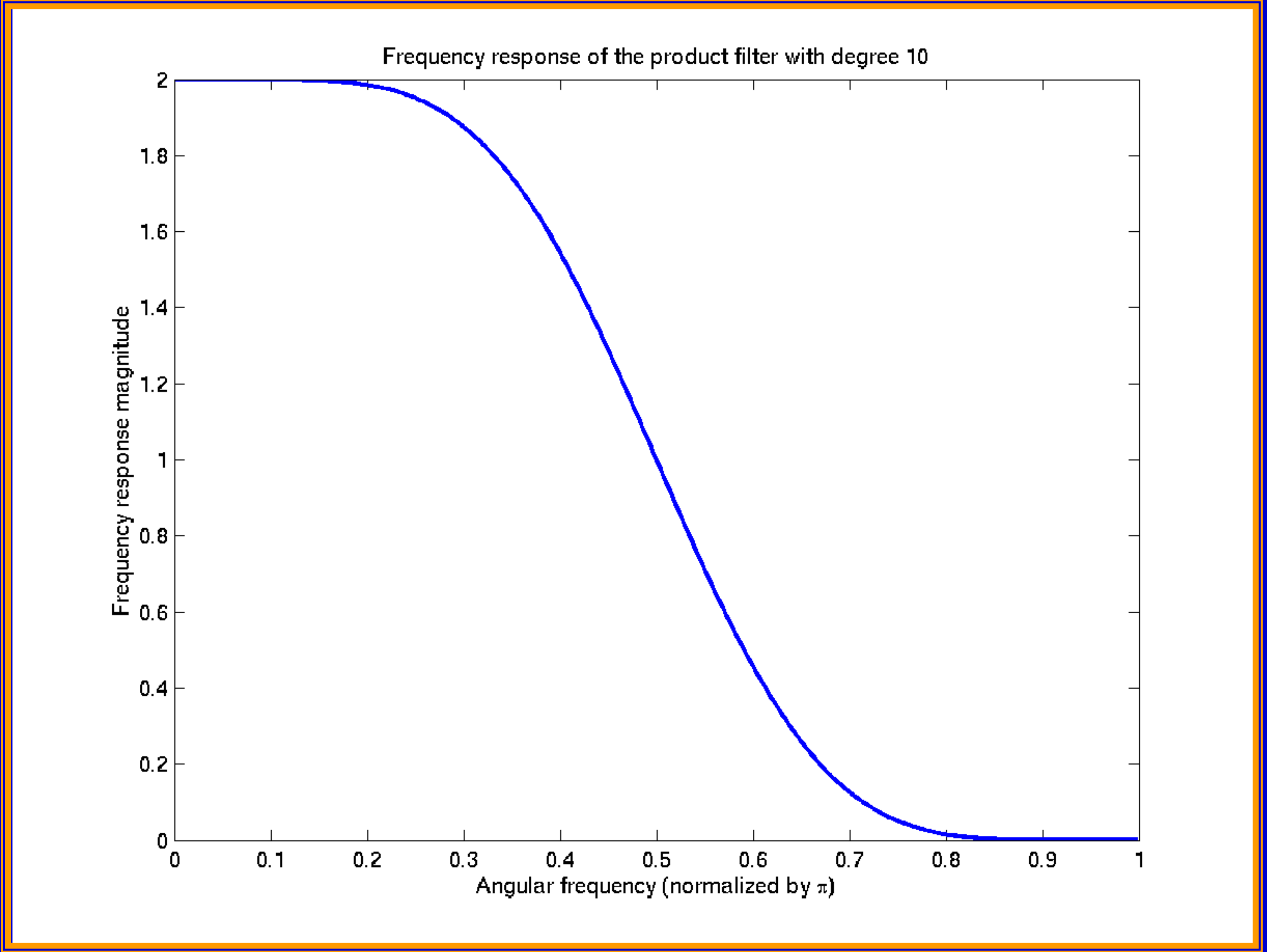
Degree-6 (p=2): Freq. response



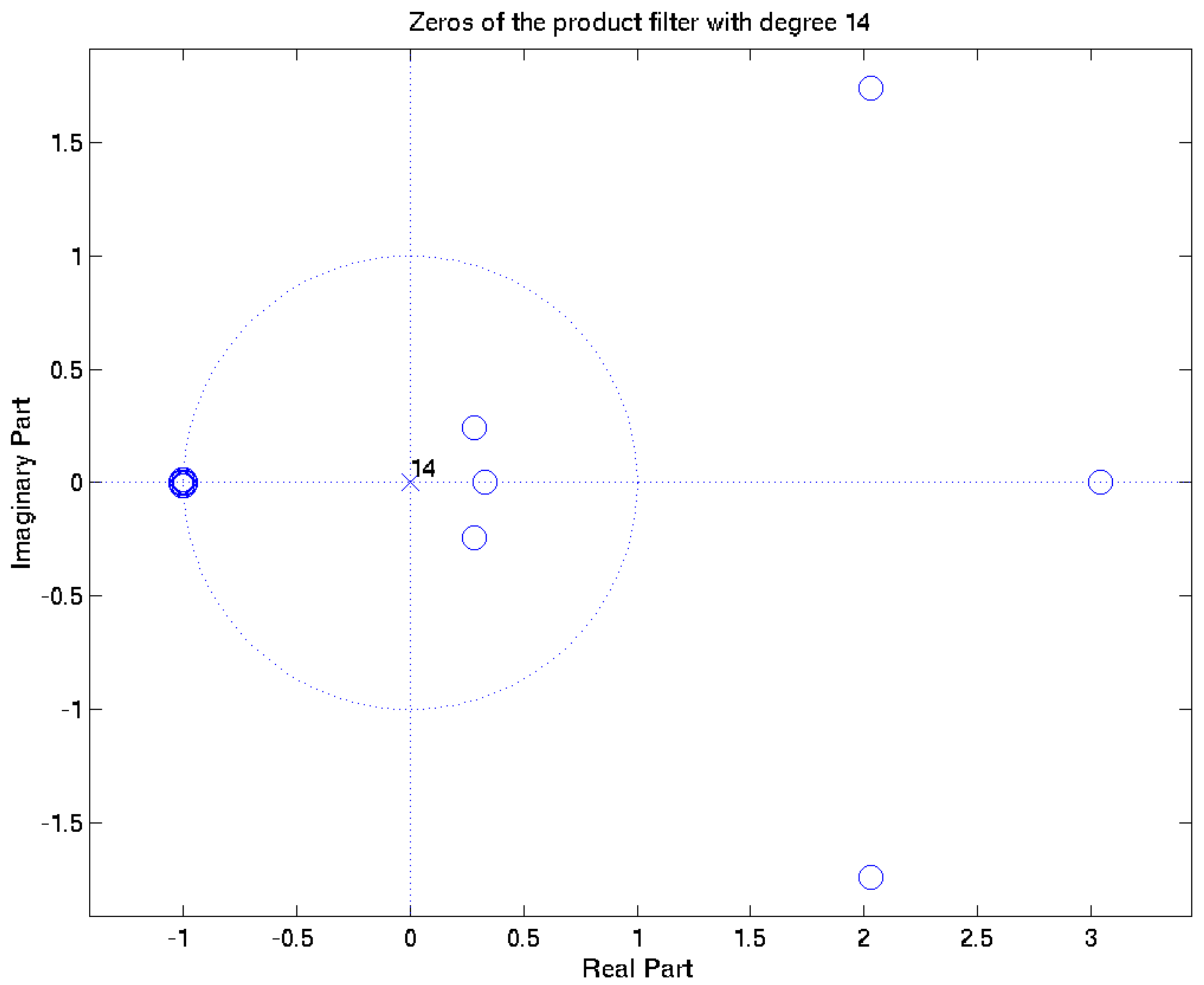
Degree-10 (p=3): pole-zero plot



Degree-10 (p=3): Freq. response



Degree-14 (p=4): pole-zero plot



Degree-14 (p=4): Freq. response