Course 18.327 and 1.130
Wavelets and Filter Banks

Modulation and Polyphase Representations:
Noble Identities;
Block Toeplitz Matrices and Block z-transforms;
Polyphase Examples

Modulation Matrix

Matrix form of PR conditions:

$$[F_0(z)\ \ F_1(z)] \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} = [\,2z^{-l}\ \ 0\,]$$

1 2 3

Modulation matrix, $H_m(z)$

So

$$[\,F_0(z)\ \ F_1(z)] = [2z^{-l}\ \ 0]\ H_m^{-1}(z)$$

$$H_m^{-1}(z) = \frac{1}{?} \begin{bmatrix} H_1(-z) & -H_0(-z) \\ -H_1(z) & H_0(z) \end{bmatrix}$$

$? = H_0(z)\,H_1(-z) - H_0(-z)\,H_1(z)$ **(must be non-zero)**

$$\Rightarrow F_0(z) = \frac{1}{?}\, 2z^{-l}\ H_1(-z)$$

$$F_1(z) = -\frac{1}{?}\, 2z^{-l}\ H_0(-z)$$

Require these to be FIR

Suppose we choose $? = 2z^{-l}$

Then

$$F_0(z) = H_1(-z)$$

$$F_1(z) = -H_0(-z)$$

Synthesis modulation matrix:

Complete the second row of matrix PR conditions by replacing z with –z:

$$\begin{bmatrix} F_0(z) & F_1(z) \\ F_0(-z) & F_1(-z) \end{bmatrix} \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} = 2 \begin{bmatrix} z^{-l} & 0 \\ 0 & (-z)^{-l} \end{bmatrix}$$

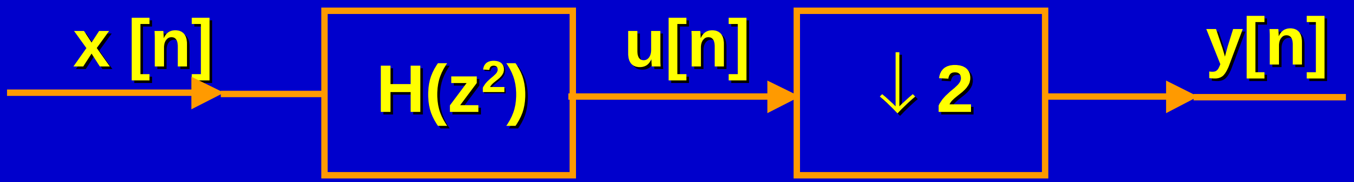
1 2 3

Synthesis modulation matrix, $F_m(z)$

Note the transpose convention in $F_m(z)$.

Noble Identities

1. Consider



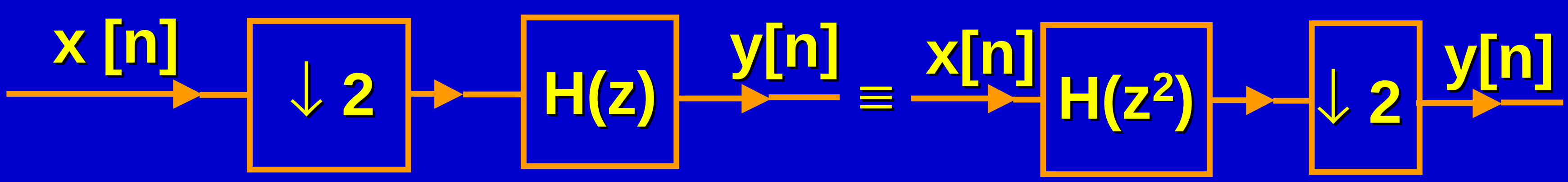
$U(z) = H(z^2)X(z)$

$Y(z) = \frac{1}{2}\{U(z^{\frac{1}{2}}) + U(-z^{\frac{1}{2}})\}$ (downsampling)

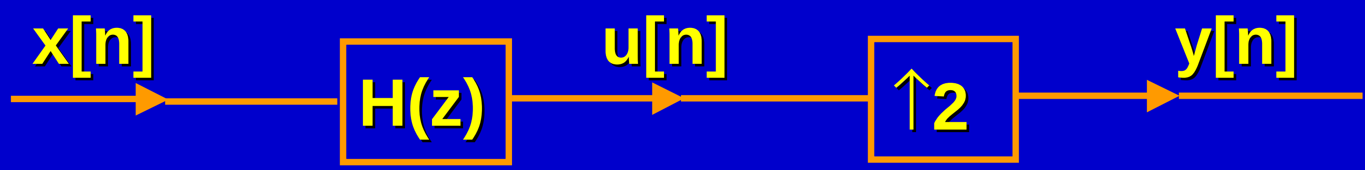
$= \frac{1}{2}\{ H(z)\, X(z^{\frac{1}{2}}) + H(z)\, X(-z^{\frac{1}{2}})\}$

$= H(z) \bullet \frac{1}{2}\{X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})\} \Rightarrow$ can downsample first

First Noble identity:



2. Consider

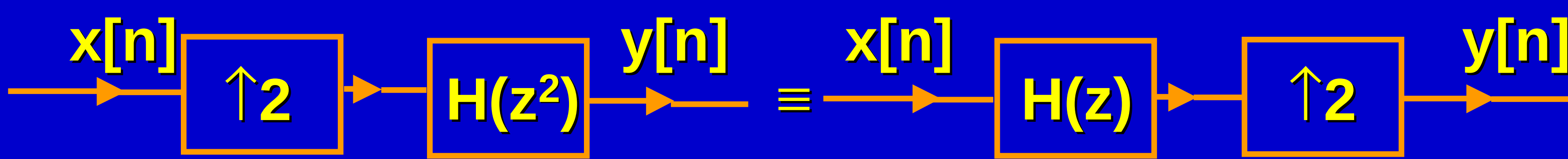


$$U(z) = H(z)\,X(z)$$

$$Y(z) = U(z^2) \qquad \text{(upsampling)}$$

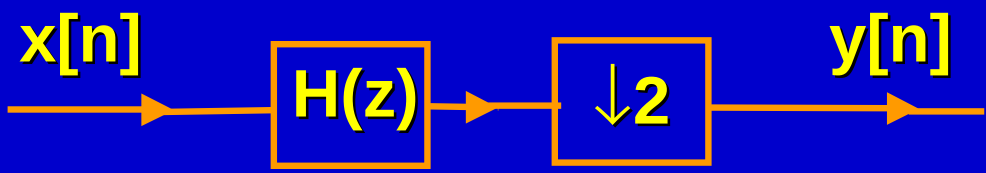
$$= H(z^2)\,X(z^2) \qquad \Rightarrow \text{can upsample first}$$

Second Noble Identity:

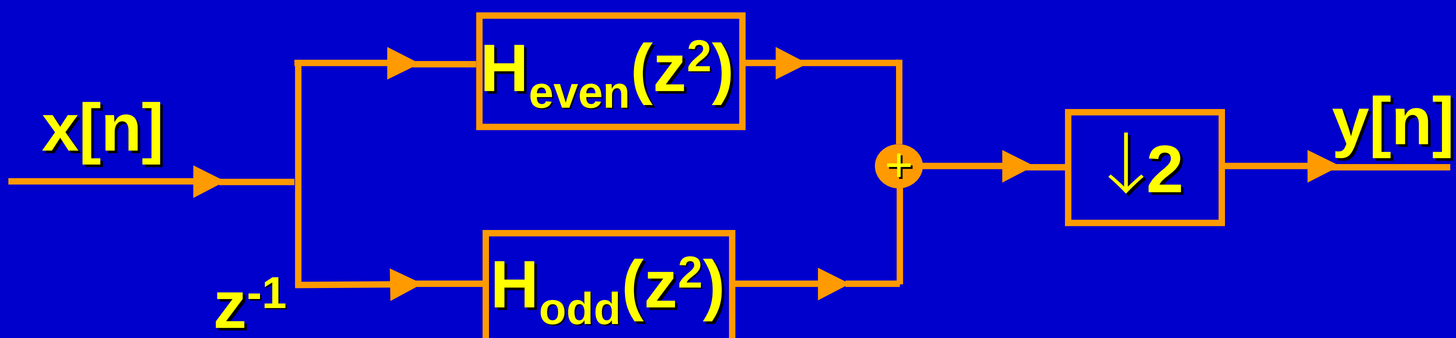


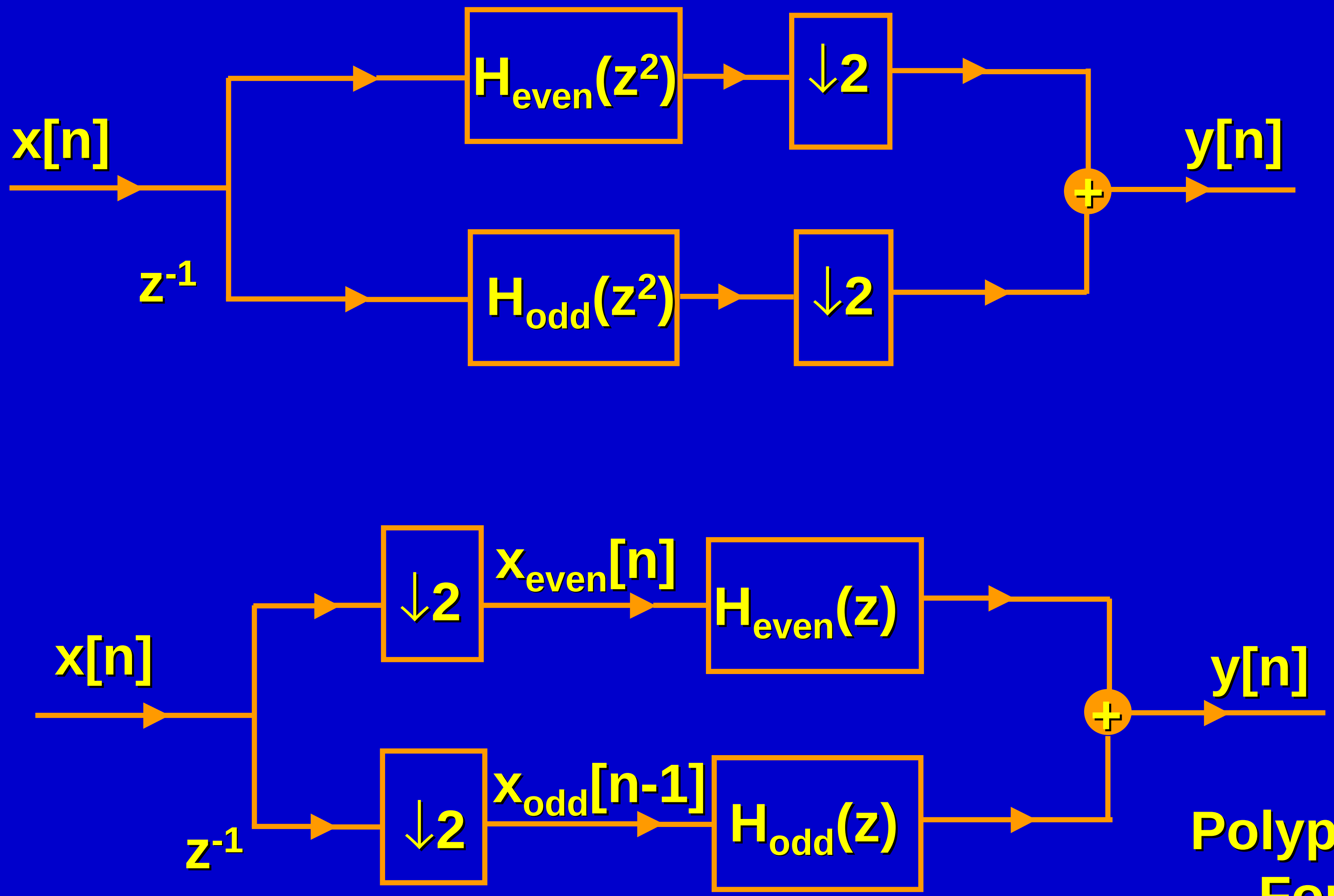
Derivation of Polyphase Form

1. Filtering and downsampling:



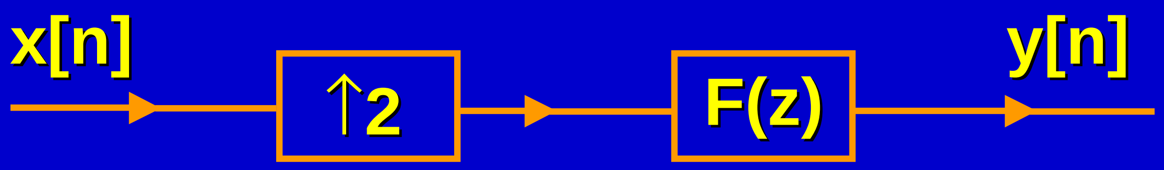
$$H(z) = H_{even}(z^2) + z^{-1} H_{odd}(z^2); \quad h_{even}[n] = h[2n]$$
$$h_{odd}[n] = h[2n+1]$$



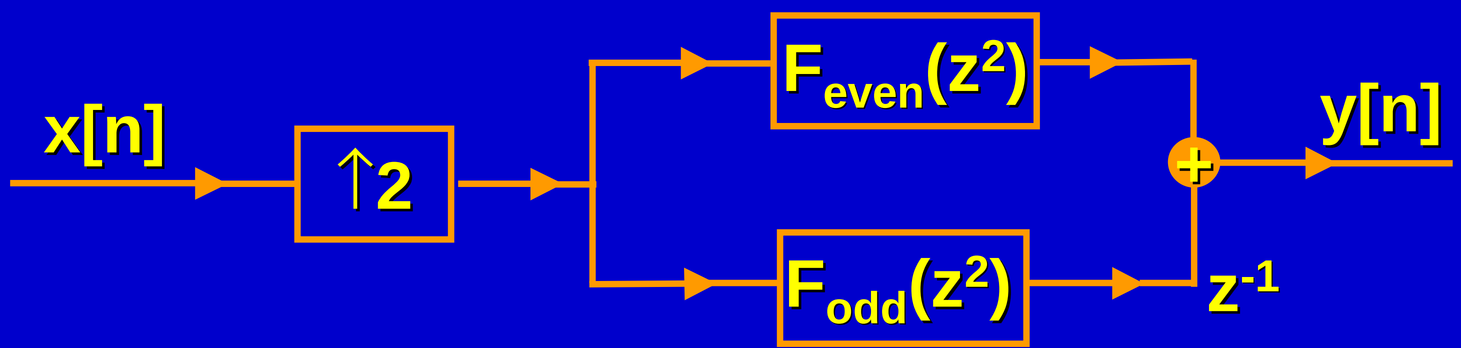


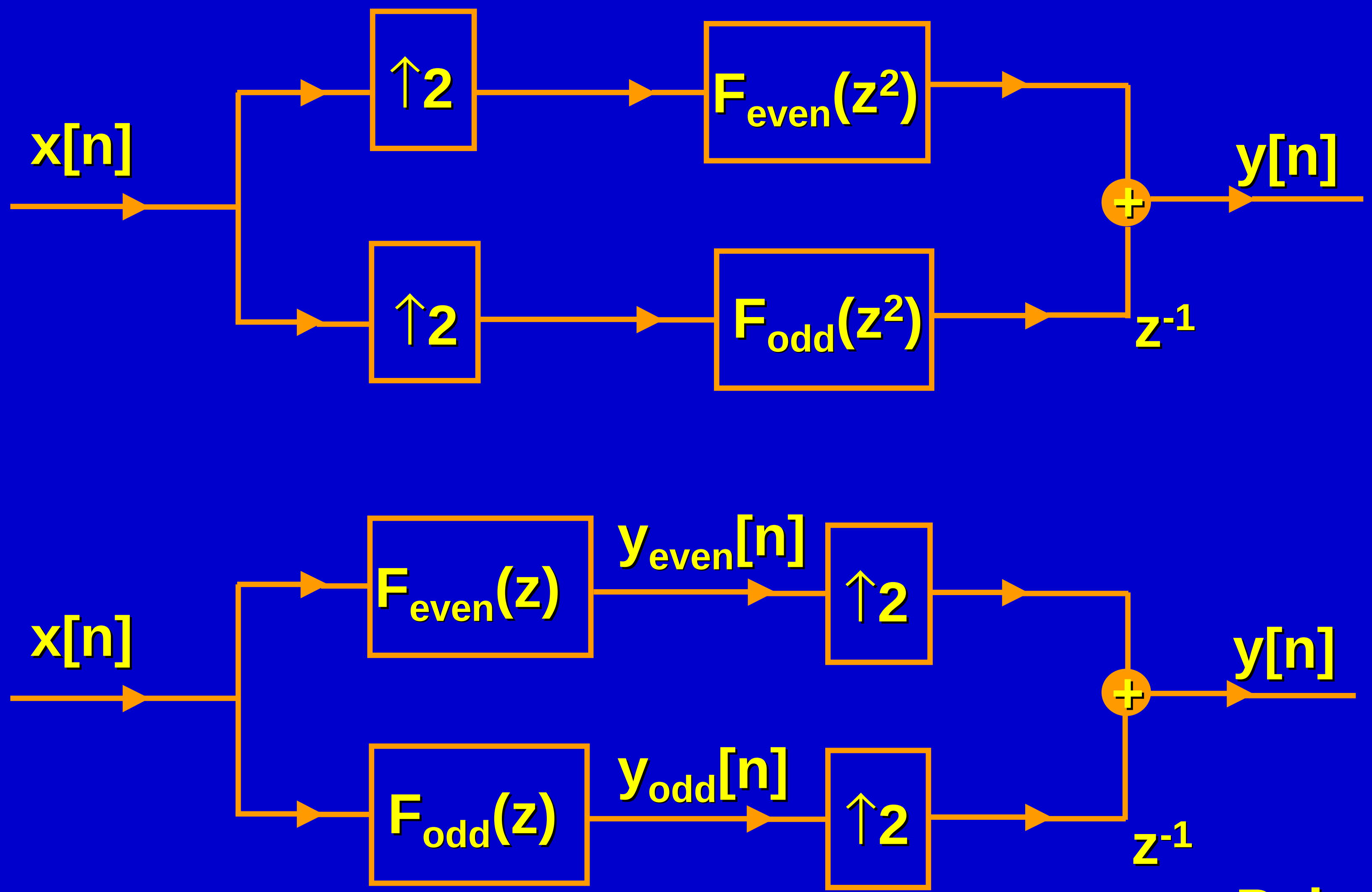
Polyphase Form

2. Upsampling and filtering



$$F(z) = F_{even}(z^2) + z^{-1}\, F_{odd}\,(z^2)$$





Polyphase Form

Polyphase Matrix

Consider the matrix corresponding to the analysis filter bank in interleaved form. This is a block Toeplitz matrix:

$$H_b = \begin{bmatrix} & & & & \vdots & & & & \\ \cdots & h_0[3] & h_0[2] & h_0[1] & h_0[0] & 0 & 0 & \cdots \\ \cdots & h_1[3] & h_1[2] & h_1[1] & h_1[0] & 0 & 0 & \cdots \\ \cdots & 0 & 0 & h_0[3] & h_0[2] & h_0[1] & h_0[0] & \cdots \\ \cdots & 0 & 0 & h_1[3] & h_1[2] & h_1[1] & h_1[0] & \cdots \\ & & & & \ddots & & & \end{bmatrix}$$

4-tap Example

Taking block z-transform we get:

$$H_p(z) = \begin{bmatrix} h_0[0] & h_0[1] \\ h_1[0] & h_1[1] \end{bmatrix} + z^{-1} \begin{bmatrix} h_0[2] & h_0[3] \\ h_1[2] & h_1[3] \end{bmatrix}$$

$$= \begin{bmatrix} h_0[0] + z^{-1} h_0[2] & h_0[1] + z^{-1} h_0[3] \\ h_1[0] + z^{-1} h_1[2] & h_1[1] + z^{-1} h_1[3] \end{bmatrix}$$

$$= \begin{bmatrix} H_{0,even}(z) & H_{0,odd}(z) \\ H_{1,even}(z) & H_{1,odd}(z) \end{bmatrix}$$

This is the polyphase matrix for a 2-channel filter bank.

Similarly, for the synthesis filter bank:

$$
F_b = \begin{bmatrix}
 & \vdots & \vdots & \vdots & \vdots & \\
 & f_0[0] & f_1[0] & 0 & 0 & \\
 & f_0[1] & f_1[1] & 0 & 0 & \\
\cdots & f_0[2] & f_1[2] & f_0[0] & f_1[0] & \cdots \\
 & f_0[3] & f_1[3] & f_0[1] & f_1[1] & \\
 & 0 & 0 & f_0[2] & f_1[2] & \\
 & 0 & 0 & f_0[3] & f_1[3] & \\
 & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix}
$$

$$F_p(z) = \begin{bmatrix} f_0[0] & f_1[0] \\ f_0[1] & f_1[1] \end{bmatrix} + z^{-1} \begin{bmatrix} f_0[2] & f_1[2] \\ f_0[3] & f_1[3] \end{bmatrix}$$

$$= \begin{bmatrix} F_{0,even}[z] & F_{1,even}[z] \\ F_{0,\,odd}[z] & F_{1,\,odd}[z] \end{bmatrix}$$

Note transpose convention for synthesis polyphase matrix

- **Perfect reconstruction condition in polyphase domain:**

$$F_p(z)\ H_p(z) = I \quad \text{(centered form)}$$

This means that $H_p(z)$ must be invertible for all z on the unit circle, i.e.

$$\det H_p(e^{i\omega}) \neq 0 \text{ for all frequencies } \omega.$$

- Given that the analysis filters are FIR, the requirement for the synthesis filters to be also FIR is:

$$\det H_p(z) = z^{-l} \quad \text{(simple delay)}$$

because $H_p^{-1}(z)$ must be a polynomial.

- Condition for orthogonality: $F_p(z)$ is the transpose of $H_p(z)$, i.e.

$$H_p^T(z^{-1})\ H_p(z) = I$$

i.e. $H_p(z)$ should be paraunitary.

Relationship between Modulation and Polyphase Matrices

$$H_0(z) = H_{0,even}(z^2) + z^{-1} H_{0,odd}(z^2)\ ; \quad \left\{ \begin{array}{l} h_{0,even}[n] = h_0[2n] \\ h_{0,odd}[n] = h_0[2n+1] \end{array} \right.$$

$$H_1(z) = H_{1,even}(z^2) + z^{-1} H_{1,odd}(z^2)$$

Two more equations by replacing z with -z.

So in matrix form:

$$\underbrace{\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}}_{H_m(z)} = \underbrace{\begin{bmatrix} H_{0,even}(z^2) & H_{0,odd}(z^2) \\ H_{1,even}(z^2) & H_{1,odd}(z^2) \end{bmatrix}}_{H_p(z^2)} \begin{bmatrix} 1 & 1 \\ z^{-1} & -z^{-1} \end{bmatrix}$$

Modulation matrix Polyphase matrix

But

$$\begin{bmatrix} 1 & 1 \\ z^{-1} & -z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & \\ & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

1 2 3 1 2 3

$D_2(z)$ — **Delay Matrix** F_2 — **2-point DFT Matrix**

$$F_N = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{N-1} \\ 1 & w^2 & w^4 & \dots & w^{2(N-1)} \\ . & . & . & & \\ . & . & . & & \\ . & . & . & & \\ 1 & w^{N-1} & w^{2(N-1)} & & w^{(N-1)^2} \end{bmatrix} ; \quad w = e^{i\frac{2\pi}{N}} \longrightarrow \text{N-point DFT Matrix}$$

$$F_N^{-1} = \frac{1}{N} \overline{F}_N$$

Complex conjugate: replace w with $\overline{w} = e^{-i\frac{2\pi}{N}}$

So, in general

$$\boxed{H_m(z)\, F_N^{-1} \quad = \quad H_p(z^N)\, D_N(z)}$$

N = # of channels in filterbank

(N = 2 in our example)

Polyphase Matrix

Example: Daubechies 4-tap filter

$$h_0[0] = \frac{1+\sqrt{3}}{4\sqrt{2}} \quad h_0[1] = \frac{3+\sqrt{3}}{4\sqrt{2}} \quad h_0[2] = \frac{3-\sqrt{3}}{4\sqrt{2}} \quad h_0[3] = \frac{1-\sqrt{3}}{4\sqrt{2}}$$

$$H_0(z) = \frac{1}{4\sqrt{2}}\{(1+\sqrt{3}) + (3+\sqrt{3})\,z^{-1} + (3-\sqrt{3})\,z^{-2} + (1-\sqrt{3})\,z^{-3}\}$$

$$H_1(z) = \frac{1}{4\sqrt{2}}\{(1-\sqrt{3}) - (3-\sqrt{3})\,z^{-1} + (3+\sqrt{3})z^{-2} - (1+\sqrt{3})z^{-3}\}$$

Time domain:

$$h_0[0]^2 + h_0[1]^2 + h_0[2]^2 + h_0[3]^2 = \frac{1}{32}\{(4 + 2\sqrt{3}) + (12 + 6\sqrt{3}) + (12 - 6\sqrt{3}) + (4 - 2\sqrt{3})\}$$

$$= 1$$

$$h_0[0]\, h_0[2] + h_0[1]\, h_0[3] = \frac{1}{32}\{(2\sqrt{3}) + (-2\sqrt{3})\}$$

$$= 0$$

i.e. filter is orthogonal to its double shifts

Polyphase Domain:

$$H_{0,even}(z) = \frac{1}{4\sqrt{2}} \{(1 + \sqrt{3}) + (3 - \sqrt{3})\, z^{-1}\}$$

$$H_{0,odd}(z) = \frac{1}{4\sqrt{2}} \{(3 + \sqrt{3}) + (1 - \sqrt{3})\, z^{-1}\}$$

$$H_{1,even}(z) = \frac{1}{4\sqrt{2}} \{(1 - \sqrt{3}) + (3 + \sqrt{3})\, z^{-1}\}$$

$$H_{1,odd}(z) = \frac{1}{4\sqrt{2}} \{ - (3 - \sqrt{3}) - (1 + \sqrt{3})\, z^{-1}\}$$

$$H_p(z) = \frac{1}{4\sqrt{2}} \underbrace{\begin{bmatrix} 1 + \sqrt{3} & 3 + \sqrt{3} \\ 1 - \sqrt{3} & -(3 - \sqrt{3}) \end{bmatrix}}_{A} + \frac{1}{4\sqrt{2}} \underbrace{\begin{bmatrix} 3 - \sqrt{3} & 1 - \sqrt{3} \\ 3 + \sqrt{3} & -(1 + \sqrt{3}) \end{bmatrix}}_{B} z^{-1}$$

1 2 3

1 2 3

$$H_p(z) = A + B\,z^{-1}$$

$$H_p^T(z^{-1})\,H_p(z) = (A^T + B^T z)(A + Bz^{-1})$$
$$= (A^TA + B^TB) + A^TBz^{-1} + B^TAz$$

$$A^TA = \frac{1}{4\sqrt{2}}\begin{bmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ 3+\sqrt{3} & -(3-\sqrt{3}) \end{bmatrix} \frac{1}{4\sqrt{2}}\begin{bmatrix} 1+\sqrt{3} & 3+\sqrt{3} \\ 1-\sqrt{3} & -(3-\sqrt{3}) \end{bmatrix}$$

$$= \frac{1}{32}\begin{bmatrix} (4+2\sqrt{3}) + (4-2\sqrt{3}) & (6+4\sqrt{3}) - (6-4\sqrt{3}) \\ (6+4\sqrt{3}) - (6-4\sqrt{3}) & (12+6\sqrt{3}) + (12-6\sqrt{3}) \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{bmatrix}$$

$$B^TB = \frac{1}{4\sqrt{2}}\begin{bmatrix} 3-\sqrt{3} & 3+\sqrt{3} \\ 1-\sqrt{3} & -(1+\sqrt{3}) \end{bmatrix} \frac{1}{4\sqrt{2}}\begin{bmatrix} 3-\sqrt{3} & 1-\sqrt{3} \\ 3+\sqrt{3} & -(1+\sqrt{3}) \end{bmatrix}$$

$$= \frac{1}{32}\begin{bmatrix} (12-6\sqrt{3})+(12+6\sqrt{3}) & (6-4\sqrt{3})-(6+4\sqrt{3}) \\ (6-4\sqrt{3})-(6+4\sqrt{3}) & (4-2\sqrt{3})+(4+2\sqrt{3}) \end{bmatrix}$$

$$= \begin{bmatrix} 3/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 1/4 \end{bmatrix}$$

$$\Rightarrow A^TA + B^TB = I$$

$$A^T B = \frac{1}{4\sqrt{2}} \begin{bmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ 3+\sqrt{3} & -(3-\sqrt{3}) \end{bmatrix} \frac{1}{4\sqrt{2}} \begin{bmatrix} 3-\sqrt{3} & 1-\sqrt{3} \\ 3+\sqrt{3} & -(1+\sqrt{3}) \end{bmatrix}$$

$$= \frac{1}{32} \begin{bmatrix} (2\sqrt{3}) + (-2\sqrt{3}) & (-2) - (-2) \\ (6) - (6) & (-2\sqrt{3}) + (2\sqrt{3}) \end{bmatrix}$$

$$= \quad 0$$

$$B^T A \quad = \quad (A^T B)^T \quad = \quad 0$$

So

$$H_p^T(z^{-1}) \; H_p(z) \; = \; I$$ i.e. $H_p(z)$ is a Paraunitary Matrix

Modulation domain:

$$H_0(z)\ H_0(z^{-1}) = P(z) = \frac{1}{16}\ (-z^3 + 9z + 16 + 9z^{-1} - z^{-3})$$

$$H_0(-z)\ H_0(-z^{-1}) = P(-z) = \frac{1}{16}\ (z^3 - 9z + 16 - 9z^{-1} + z^{-3})$$

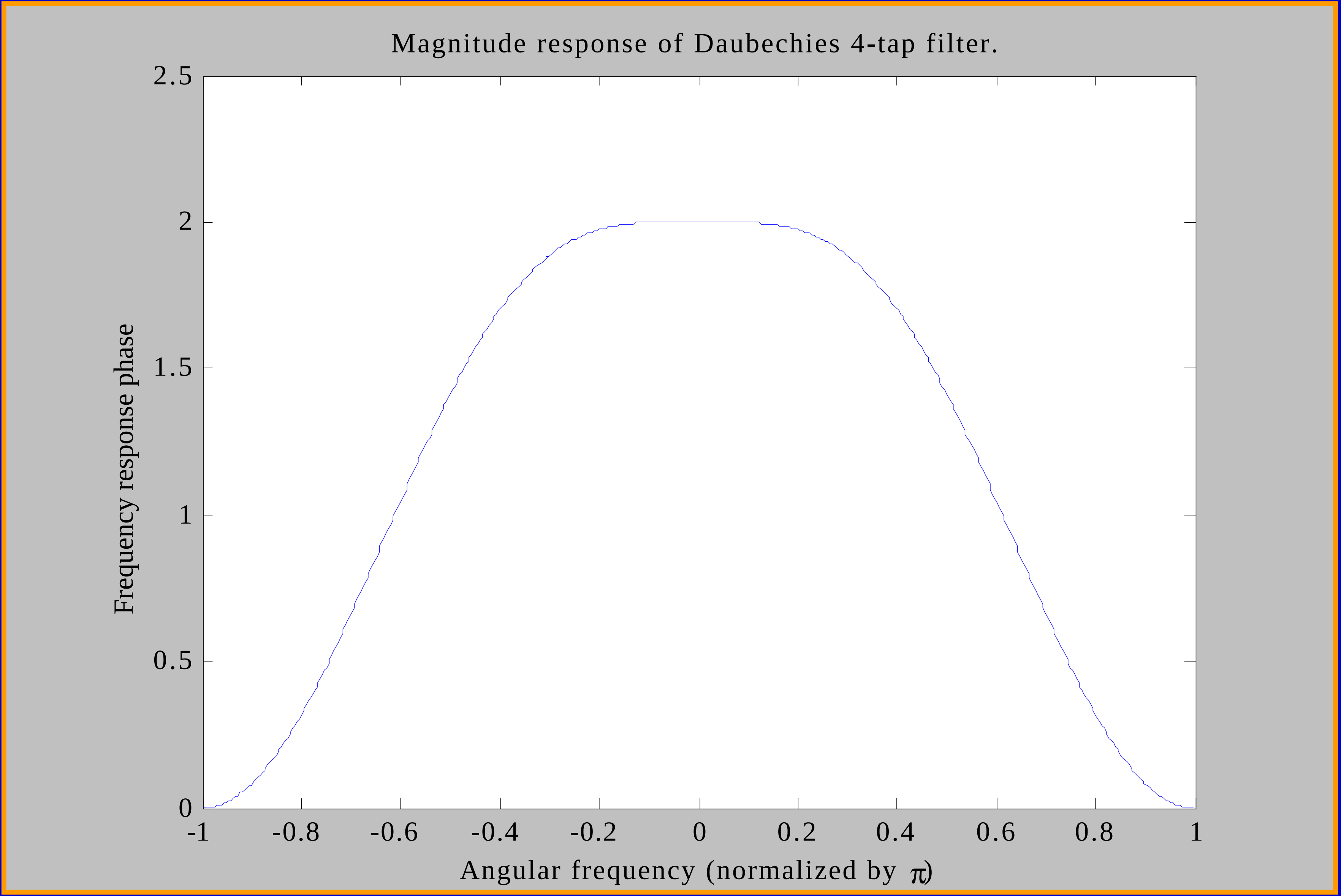
So

$$H_0(z)\ H_0(z^{-1}) + H_0(-z)\ H_0(-z^{-1}) = 2$$

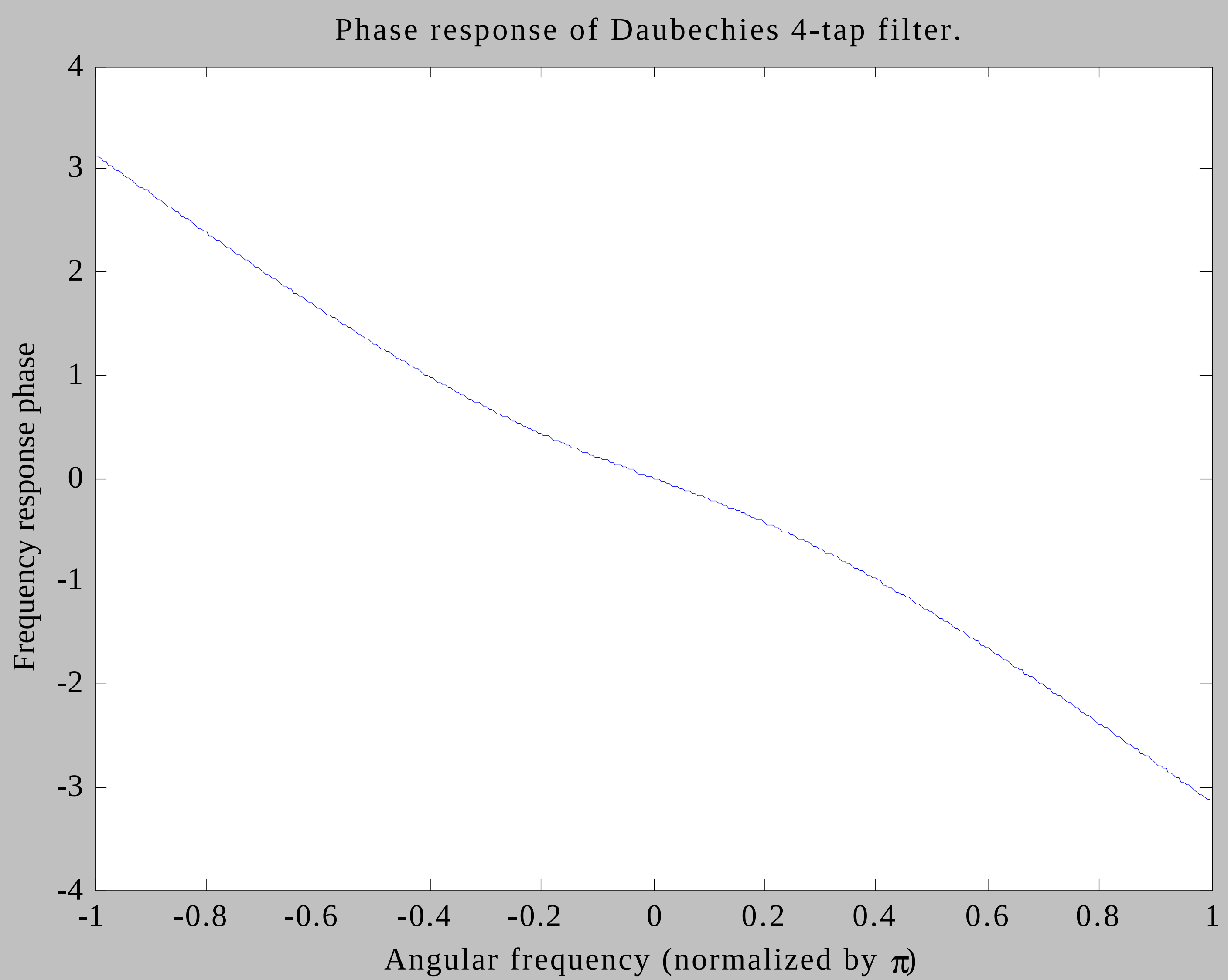
i.e.

$$|H_0(\omega)|\hat{}2 + |H_0(\omega+\pi)|\hat{}2 = 2$$

Magnitude Response of Daubechies 4-tap filter.



Phase response of Daubechies 4-tap filter.



Course 18.327 and 1.130
Wavelets and Filter Banks

Orthogonal Filter Banks;
Paraunitary Matrices;
Orthogonality Condition (Condition O) in the Time Domain, Modulation Domain and Polyphase Domain

Unitary Matrices

The constant complex matrix A is said to be unitary if

$A^{\dagger} A = I$

example:

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} \qquad A^* = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{\sqrt{2}} \begin{bmatrix} -1 & i \\ -i & 1 \end{bmatrix} \qquad A^{\dagger} = A^{*T} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}$$

$\Rightarrow \; A^{\dagger} = A^{-1}$

Paraunitary Matrices

The matrix function H(z) is said to be paraunitary if it is unitary for all values of the parameter z

$$H^T(z^{-1})\ \ H(z)\ =\ I \qquad\qquad \text{for all } z \neq 0 \text{ ------------------(1)}$$

Frequency Domain:

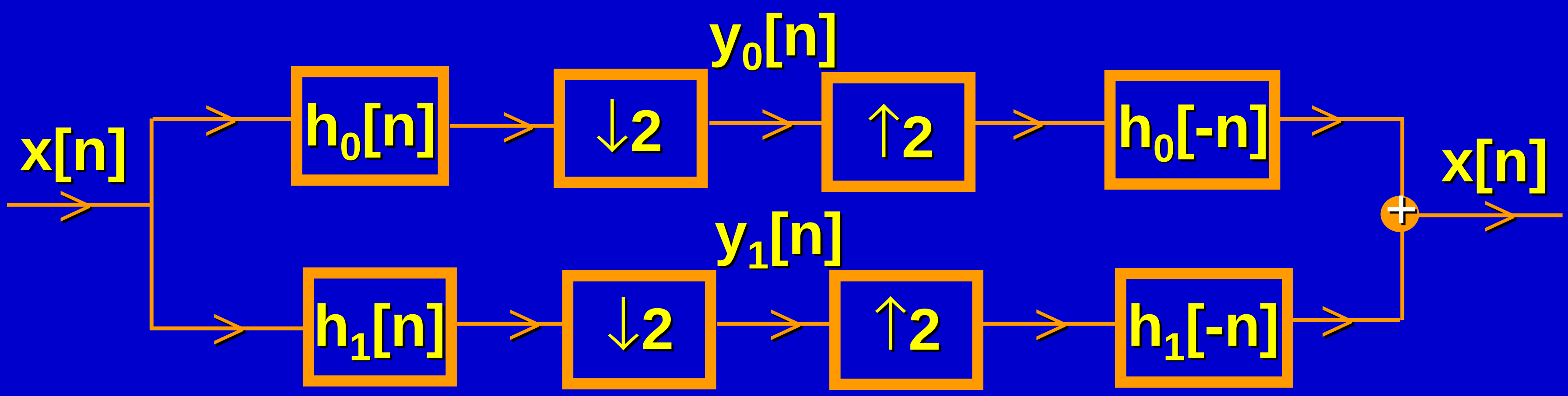
$$H^T(-\omega)\ \ H(\omega)\ =\ I \qquad\qquad \text{for all } \omega$$

or $H^{*T}(\omega)\ \ H(\omega) = I$

Note: we are assuming that h[n] are real.

Orthogonal Filter Banks

Centered form (PR with no delay):



Synthesis bank = transpose of analysis bank

$h_0[n]$ causal $\Rightarrow f_0[n] \equiv h_0[-n]$ anticausal

What are the conditions on $h_0[n]$, $h_1[n]$, in the

(i) time domain?
(ii) polyphase domain?
(iii) modulation domain?

Time Domain

Analysis: N = 3 (filter length = 4)

$$
\begin{bmatrix} \text{M} \\ y_0[0] \\ y_0[1] \\ y_0[2] \\ y_0[3] \\ \text{M} \\ \hline \text{M} \\ y_1[0] \\ y_1[1] \\ y_1[2] \\ y_1[3] \\ \text{M} \end{bmatrix}
=
\begin{bmatrix}
\text{K} & & & & & & & & & \\
h_0[3] & h_0[2] & h_0[1] & h_0[0] & & & & & & \\
 & & h_0[3] & h_0[2] & h_0[1] & h_0[0] & & & & \\
 & & & & h_0[3] & h_0[2] & h_0[1] & h_0[0] & & \\
 & & & & & & h_0[3] & h_0[2] & h_0[1] & h_0[0] \\
 & & & & & & & & & \text{L} \\
\hline
\text{K} & & & & & & & & & \\
h_1[3] & h_1[2] & h_1[1] & h_1[0] & & & & & & \\
 & & h_1[3] & h_1[2] & h_1[1] & h_1[0] & & & & \\
 & & & & h_1[3] & h_1[2] & h_1[1] & h_1[0] & & \\
 & & & & & & h_1[3] & h_1[2] & h_1[1] & h_1[0] \\
 & & & & & & & & & \text{L}
\end{bmatrix}
\begin{bmatrix} \text{M} \\ x[-3] \\ x[-2] \\ x[-1] \\ x[-0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ \text{M} \end{bmatrix}
\quad \text{-----(2)}
$$

1 2 3

W

Synthesis:

$$
\begin{bmatrix} \vdots \\ x[-3] \\ x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ \vdots \end{bmatrix}
=
\begin{bmatrix}
\vdots & & & & \vdots & & & \\
h_0[3] & & & & h_1[3] & & & \\
h_0[2] & & & & h_1[2] & & & \\
h_0[1] & h_0[3] & & & h_1[1] & h_1[3] & & \\
h_0[0] & h_0[2] & & & h_1[0] & h_1[2] & & \\
 & h_0[1] & h_0[3] & & & h_1[1] & h_1[3] & \\
 & h_0[0] & h_0[2] & & & h_1[0] & h_1[2] & \\
 & & h_0[1] & h_0[3] & & & h_1[1] & h_1[3] \\
 & & h_0[0] & h_0[2] & & & h_1[0] & h_1[2] \\
 & & & h_0[1] & & & & h_1[1] \\
 & & & h_0[0] & & & & h_1[0] \\
 & & & & \vdots & & & \vdots
\end{bmatrix}
\begin{bmatrix} \vdots \\ y_0[0] \\ y_0[1] \\ y_0[2] \\ y_0[3] \\ \vdots \\ \hline \vdots \\ y_1[0] \\ y_1[1] \\ y_1[2] \\ y_1[3] \\ \vdots \end{bmatrix}
$$

1 2 3

-----(3)

W^T

Orthogonality condition (Condition O) is

$$W^TW = I = WW^T \Rightarrow W \text{ orthogonal matrix}$$

Block Form:

$$W = \begin{bmatrix} L \\ B \end{bmatrix}$$

$$L^TL + B^TB = I$$

$$\begin{bmatrix} LL^T & LB^T \\ BL^T & BB^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$LL^T = I \Rightarrow \sum_n h_0[n]\, h_0[n-2k] = \delta[k] \quad \text{--------------(4)}$$

$$LB^T = 0 \Rightarrow \sum_n h_0[n]\, h_1[n-2k] = 0 \quad \text{--------------(5)}$$

$$BB^T = I \Rightarrow \sum_n h_1[n]\, h_1[n-2k] = \delta[k] \quad \text{--------------(6)}$$

Good choice for $h_1[n]$:

$$h_1[n] = (-1)^n h_0[N-n] \quad ; \text{ N odd } \text{--------------(7)}$$

$\longrightarrow$ Alternating flip

Example: N = 3

$$\begin{aligned} h_1[0] &= h_0[3] \\ h_1[1] &= -h_0[2] \\ h_1[2] &= h_0[1] \\ h_1[3] &= -h_0[0] \end{aligned}$$

With this choice, Equation (5) is automatically satisfied:

$k = -1$: $h_0[0]h_0[1] - h_0[1]h_0[0] = 0$

$k = 0$: $h_0[0]h_0[3] - h_0[1]h_0[2] + h_0[2]h_0[1] - h_0[3]h_0[0] = 0$

$k = 1$: $h_0[2]h_0[3] - h_0[3]h_0[2] = 0$

$k = \pm 2$: no overlap

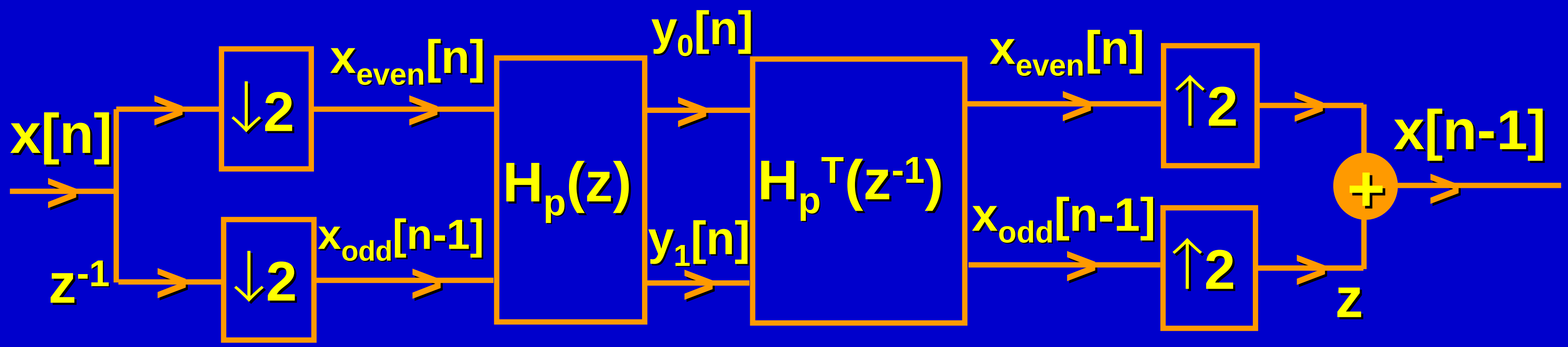
Also, Equation (6) reduces to Equation (4)

$$\delta[k] = \sum_n h_1[n]\, h_1[n-2k] = \sum_n (-1)^n\, h_0[N-n]\, (-1)^{n-2k}\, h_0[N-n+2k]$$

$$= \sum_l h_0[l]\, h_0[l + 2k]$$

So, Condition O on the lowpass filter + alternating flip for highpass filter lead to orthogonality

Polyphase Domain



$$H_p(z) = \begin{bmatrix} H_{0,even}(z) & H_{0,odd}(z) \\ H_{1,even}(z) & H_{1,odd}(z) \end{bmatrix}$$

Polyphase Matrix

Condition O:

$$H_p^T(z^{-1})\ H_p(z) = I \Rightarrow H_p(z) \text{ is paraunitary}$$

$$\begin{bmatrix} H_{0,even}(z^{-1}) & H_{1,even}(z^{-1}) \\ H_{0,odd}(z^{-1}) & H_{1,odd}(z^{-1}) \end{bmatrix} \begin{bmatrix} H_{0,even}(z) & H_{0,odd}(z) \\ H_{1,even}(z) & H_{1,odd}(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reverse the order of multiplication:

$$\begin{bmatrix} H_{0,even}(z) & H_{0,odd}(z) \\ H_{1,even}(z) & H_{1,odd}(z) \end{bmatrix} \begin{bmatrix} H_{0,even}(z^{-1}) & H_{1,even}(z^{-1}) \\ H_{0,odd}(z^{-1}) & H_{1,odd}(z^{-1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Express Condition O as a condition on $H_{0,even}(z)$, $H_{0,odd}(z)$:

$$H_{0,even}(z)\ H_{0,even}(z^{-1}) + H_{0,odd}(z)\ H_{0,odd}(z^{-1}) = 1 \qquad \text{------}(8)$$

Frequency domain:

$$|H_{0,even}(\omega)|^2 + |H_{0,odd}(\omega)|^2 = 1 \qquad \text{-------}(9)$$

The alternating flip construction for $H_1(z)$ ensures that the remaining conditions are satisfied.

$$H_0(z) = H_{0,even}(z^2) \; + \; z^{-1} H_{0,odd}(z^2)$$

$$H_1(z) = -z^{-N} H_0(-z^{-1}) \qquad \text{alternating flip}$$

$$= -z^{-N} \{H_{0,even}(z^{-2}) \; - \; z\, H_{0,odd}(z^{-2})\}$$

$$= \underbrace{-z^{-N} H_{0,even}(z^{-2})}_{z^{-1} H_{1,odd}(z^2)} \; + \; \underbrace{z^{-N+1} H_{0,odd}(z^{-2})}_{H_{1,even}(z^2)}$$

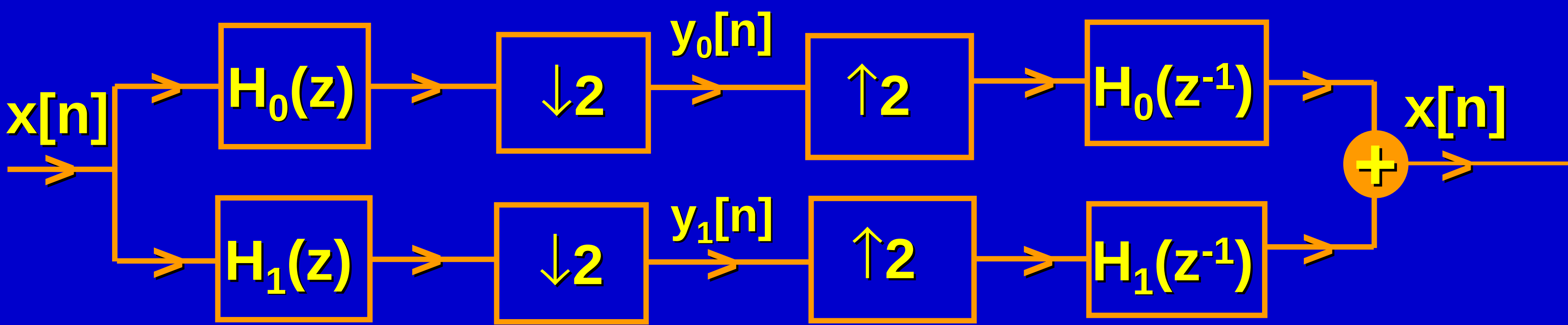
So

$$H_{1,even}(z) \; = \; z^{(-N+1)/2} H_{0,odd}(z^{-1})$$
$$H_{1,odd}(z) \; = -z^{(-N+1)/2} H_{0,even}(z^{-1})$$

$\Rightarrow$ $\quad H_{0,even}(z) H_{1,even}(z^{-1}) + H_{0,odd}(z) H_{1,odd}(z^{-1}) \; = 0$

and $\quad H_{1,even}(z) H_{1,even}(z^{-1}) + H_{1,odd}(z) H_{1,odd}(z^{-1}) \; = 1$

Modulation Domain



PR conditions:

$H_0(z)\,H_0(z^{-1}) + H_1(z)\,H_1(z^{-1}) = 2$ --------(10) No distortion

$H_0(-z)\,H_0(z^{-1}) + H_1(-z)\,H_1(z^{-1}) = 0$ --------(11) Alias cancellation

$$[H_0(z^{-1}) \quad H_1(z^{-1})] \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} = \begin{bmatrix} 2 & 0 \end{bmatrix}$$

1 2 3

$H_m(z)$ modulation matrix

Replace z with –z in Equations (10) and (11)

$$H_0(-z)\,H_0(-z^{-1}) \;+\; H_1(-z)\,H_1(-z^{-1}) \;=\; 2$$

$$H_0(z)\,H_0(-z^{-1}) \;+\; H_1(z)\,H_1(-z^{-1}) \;=\; 0$$

$$\underbrace{\begin{bmatrix} H_0(z^{-1}) & H_1(z^{-1}) \\ H_0(-z^{-1}) & H_1(-z^{-1}) \end{bmatrix}}_{H_m^T(z^{-1})} \underbrace{\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}}_{H_m(z)} = \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}_{2I}$$

1 2 3 1 2 3 1 2 3

Condition O:

$$H_m^T(z^{-1})\;H_m(z) \;=\; 2I \;\Rightarrow\; H_m(z) \text{ is paraunitary}$$

Reverse the order of multiplication:

$$\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} H_0(z^{-1}) & H_1(z^{-1}) \\ H_0(-z^{-1}) & H_1(-z^{-1}) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Express Condition O as a condition on $H_0(z)$:

$$\boxed{H_0(z)\, H_0(z^{-1}) + H_0(-z)\, H_0(-z^{-1}) = 2} \text{ ------------(12)}$$

Frequency Domain:

$$\boxed{|H_0(\omega)|^2 + |H_0(\omega + \pi)|^2 = 2} \text{ ------------(13)}$$

Again, the remaining conditions are automatically satisfied by the alternating flip choice, $H_1(z) = -z^{-N} H_0(-z^{-1})$

Summary

Condition O as a constraint on the lowpass filter:

- **Matrix form:** $LL^T = I$

- **Coefficient form:** $\sum_n h[n]h[n-2k] = \delta[k]$

- **Polyphase form:**

$$H_{0,even}(z)\, H_{0,even}(z^{-1}) = H_{0,odd}(z)\, H_{0,odd}(z^{-1}) = 1$$

- **Modulation form:** $H_0(z)\, H_0(z^{-1}) + H_0(-z)\, H_0(-z^{-1}) = 2$

Then choose $H_1(z) = -z^{-N} H_0(-z^{-1})$; N odd

i.e., $h_1[n] = (-1)^n h_0[N-n]$

Course 18.327 and 1.130
Wavelets and Filter Banks

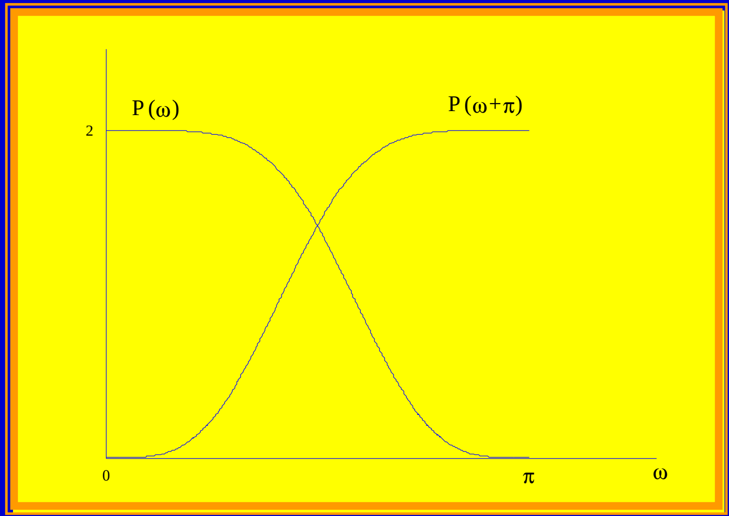
Maxflat Filters: Daubechies and Meyer Formulas.
Spectral Factorization

Formulas for the Product Filter

Halfband condition:

$$P(\omega) + P(\omega + \pi) = 2$$

Also want $P(\omega)$ to be lowpass and $p[n]$ to be symmetric.



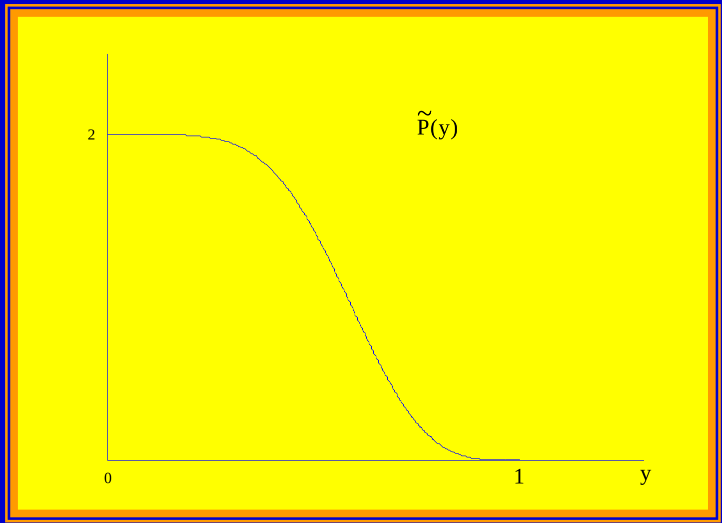
Daubechies' Approach

Design a polynomial, $\tilde{P}(y)$, of degree 2p - 1, such that

$\tilde{P}(0) = 2$

$\tilde{P}^{(l)}(0) = 0; \; l = 1, 2, \ldots, p - 1$

$\tilde{P}^{(l)}(1) = 0; \; l = 0, 1, \ldots, p - 1$



Can achieve required flatness at $y = 1$ by including a term of the form $(1 - y)^p$ i.e.

$\tilde{P}(y) = 2(1 - y)^p \, B_p(y)$

Where $B_p(y)$ is a polynomial of degree $p - 1$.

How to choose $B_p(y)$?

Let $B_p(y)$ be the binomial series expansion for $(1 - y)^{-p}$, truncated after p terms:

$$B_p(y) = 1 + py + \frac{p(p+1)}{2} y^2 + \ldots + \binom{2p-2}{p-1} y^{p-1}$$

$$= (1 - y)^{-p} + O(y^p)$$

< Higher order terms

$$(1 - y)^{-1} = \sum_{k=0}^{\infty} y^k$$

$$(1 - y)^{-p} = \sum_{k=0}^{\infty} \binom{p + k - 1}{k} y^k$$

$$|y| < 1$$

Then

$$\tilde{P}(y) = 2(1 - y)^p[(1-y)^{-p} + O(y^p)]$$
$$= 2 + O(y^p)$$

Thus

$$P^{(l)}(0) = 0 \;;\; l = 1, 2, \ldots, p\text{-}1$$

So we have

$$\tilde{P}(y) = 2\,(1-y)^p \sum_{k=0}^{p-1} \binom{p+k-1}{k} y^k$$

Now let

$$y = \left(\frac{1 - e^{i\omega}}{2}\right)\left(\frac{1 - e^{-i\omega}}{2}\right) \quad \text{maintains symmetry}$$

$$= \frac{1 - \cos\omega}{2}$$

Thus

$$\boxed{\begin{aligned} P(\omega) &= \tilde{P}\left(\frac{1-\cos\omega}{2}\right) \\ &= 2\left(\frac{1+\cos\omega}{2}\right)^p \sum_{k=0}^{p-1} \binom{p+k+1}{k} \left(\frac{1-\cos\omega}{2}\right)^k \end{aligned}}$$

z domain:

$$P(z) = 2\left(\frac{1+z}{2}\right)^p \left(\frac{1+z^{-1}}{2}\right)^p \sum_{k=0}^{p-1} \binom{p+k-1}{k} \left(\frac{1-z}{2}\right)^k \left(\frac{1-z^{-1}}{2}\right)^k$$

Meyer's Approach

Work with derivative of $\tilde{P}(y)$:

$$\tilde{P}'(y) = -C' y^{p-1}(1-y)^{p-1}$$

So

$$\tilde{P}(y) = 2 - C'\int_0^y y^{p-1}(1-y)^{p-1}\,dy \qquad (\tilde{P}(0) = 2)$$

Then

$$P(\omega) = 2 - C'\int_0^\omega \left(\frac{1-\cos\omega}{2}\right)^{p-1}\left(\frac{1+\cos\omega}{2}\right)^{p-1}\frac{\sin\omega}{2}\,d\omega$$

$$= 2 - C'\int_0^\omega \left(\frac{1-\cos^2\omega}{2}\right)^{p-1}\frac{\sin\omega}{2}\,d\omega$$

i.e. $$\boxed{P(\omega) = 2 - C\int_0^\omega \sin^{2p-1}\omega\,d\omega}$$

Spectral Factorization

Recall the halfband condition for orthogonal filters:

z domain:

$$H_0(z)\,H_0(z^{-1}) \; + \; H_0(-z)\,H_0(-z^{-1}) \; = \; 2$$

Frequency domain:

$$\left| H_0(\omega) \right|^2 + \left| H_0(\omega+\pi) \right|^2 \quad = \; 2$$

The product filter for the orthogonal case is

$$P(z) \; = \; H_0(z)\,H_0(z^{-1})$$

$$P(\omega) = \left| H_0(\omega) \right|^2 \qquad \Rightarrow P(\omega) \geq 0$$

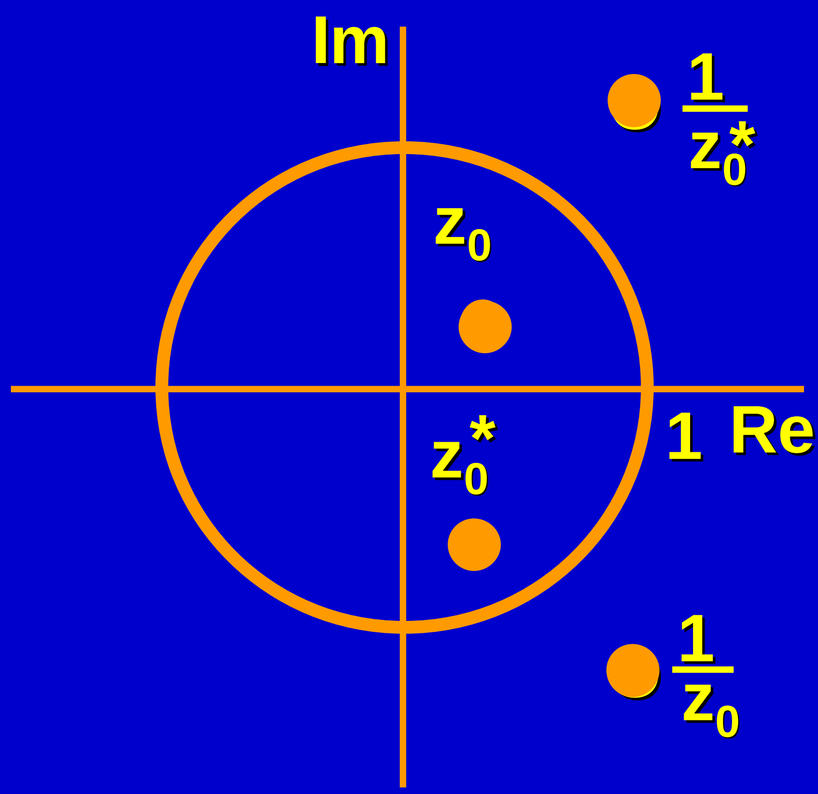
$$p[n] \; = \; h_0[n] * h_0[-n] \qquad \Rightarrow p[n] = p[-n]$$

The spectral factorization problem is the problem of finding $H_0(z)$ once $P(z)$ is known.

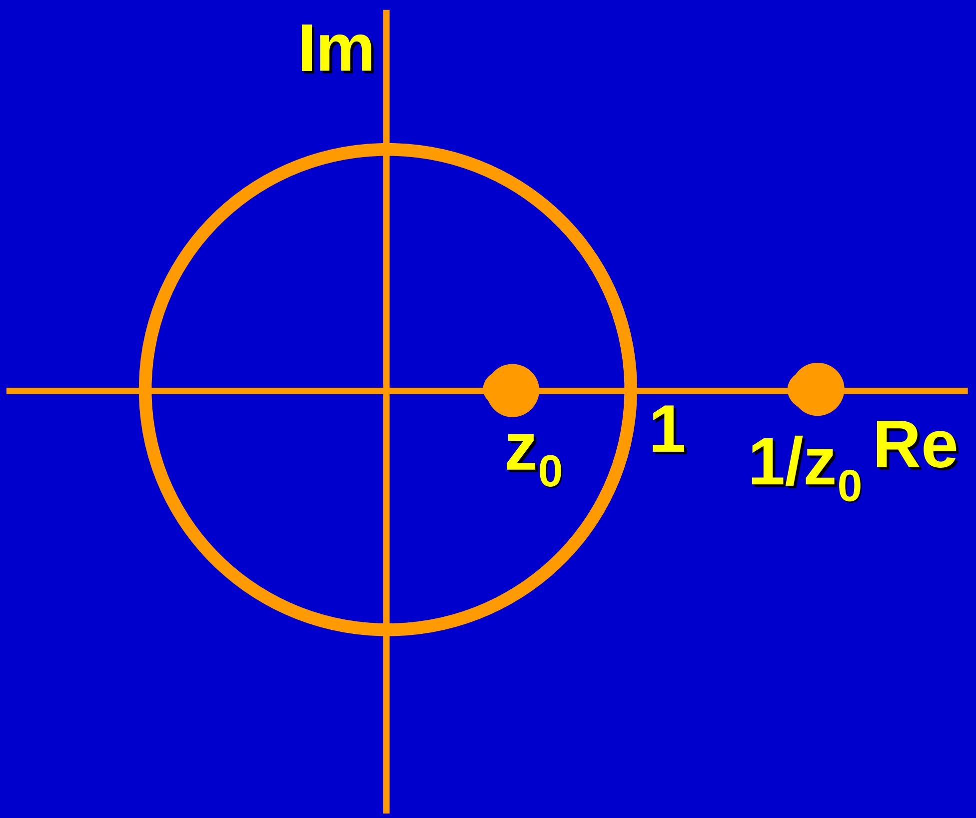
Consider the distribution of the zeros (roots) of $P(z)$.

- **Symmetry of $p[n]$ $\Rightarrow$ $P(z) = P(z^{-1})$**
  If z_0 is a root then so is z_0^{-1}.
- **If $p[n]$ are real, then the roots appear in complex, conjugate pairs.**

$$(1 - z_0 z^{-1})(1 - z_0^* z^{-1}) = 1 - \underbrace{(z_0 + z_0^*)}_{\text{real}} z^{-1} + \underbrace{(z_0 z_0^*)}_{\text{real}} z^{-2}$$



Complex zeros



Real zeros

If the zero z_0 is grouped into the spectral factor $H_0(z)$, then the zero $1/z_0$ must be grouped into $H_0(z^{-1})$.

$\Rightarrow$ $h_0[n]$ cannot be symmetric.

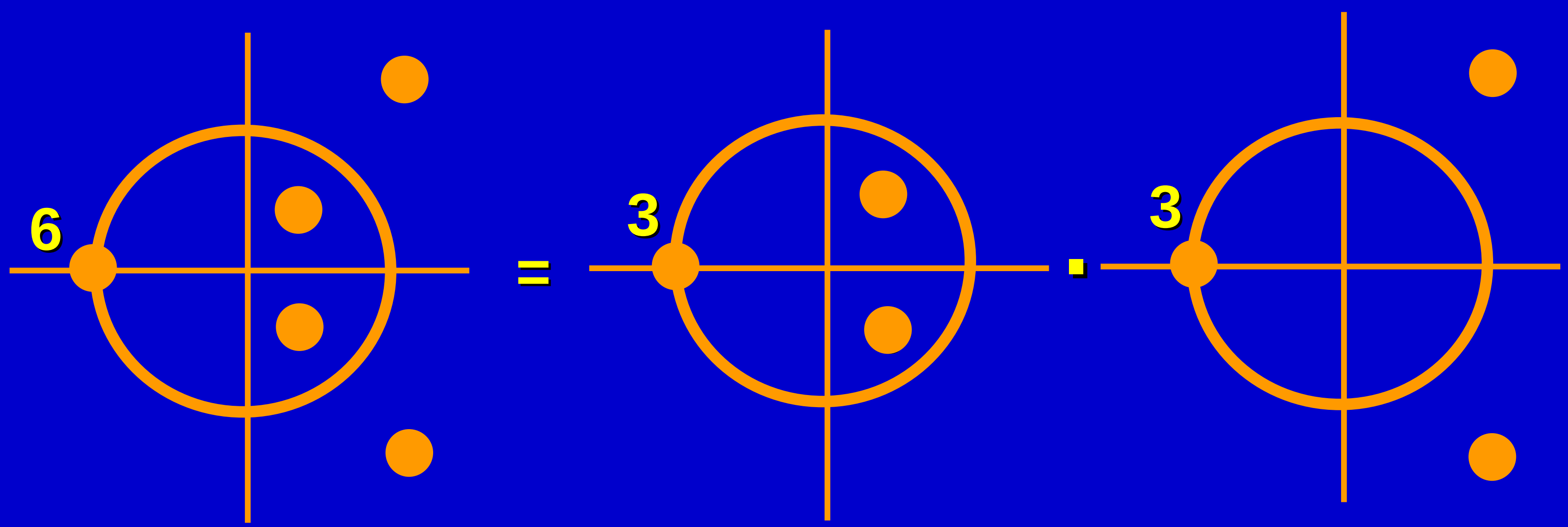
Daubechies' choice: Choose $H_0(z)$ such that

(i) all its zeros are inside or on the unit circle.

(ii) it is causal.

i.e. $H_0(z)$ is a minimum phase filter.

Example:



$P(z)$ $H_0(z)$ (Minimum phase) $H_0(z^{-1})$ (Maximum phase)

Practical Algorithms:

1. Direct Method: compute the roots of $P(z)$ numerically.
2. Cepstral Method:
   First factor out the zeros which lie on the unit circle

$$P(z) = [(1 + z^{-1})(1 + z)]^p \, Q(z)$$

Now we need to factor $Q(z)$ into $R(z)\,R(z^{-1})$ such that

i. $R(z)$ has all its zeros inside the unit circle.

ii. $R(z)$ is causal.

Then use logarithms to change multiplication into addition:

$$Q(z) = R(z) \bullet R(z^{-1})$$

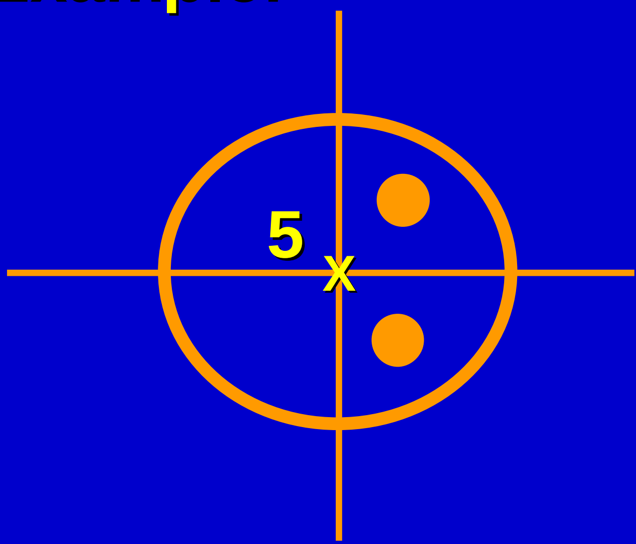
$$\underbrace{\ln Q(z)}_{\hat{Q}(z)} = \underbrace{\ln R(z)}_{\hat{R}(z)} + \underbrace{\ln R(z^{-1})}_{\hat{R}(z^{-1})}$$

Take inverse z transforms:

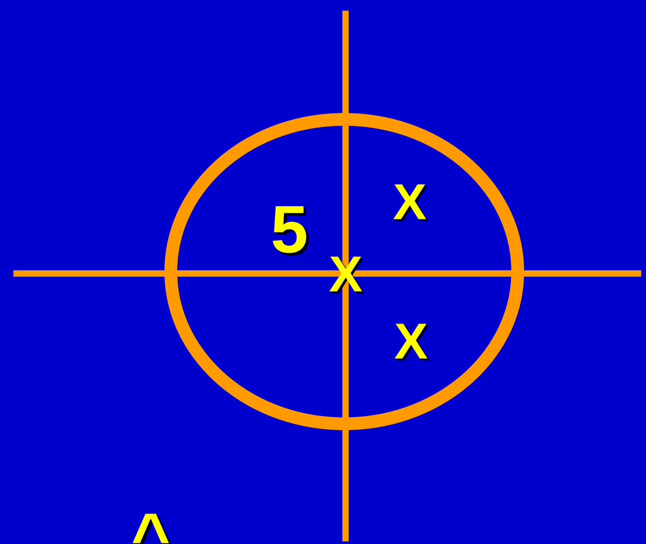
$$\hat{q}[n] = \hat{r}[n] + \hat{r}[-n]$$

Complex cepstrum of q[n]

Example:



$R(z)$



$\hat{R}(z) = \ln R(z)$

$R(z)$ has all its zeros and all its poles inside the unit circle, so $\hat{R}(z)$ has all its singularities inside the unit circle. ($\ln 0 = -\infty$, $\ln \infty = \infty$.)

All singularities inside the unit circle leads to a causal sequence, e.g.

$$X(z) = \frac{1}{1 - z_k\, z^{-1}} \qquad \text{Pole at } z = z_k$$

$$X(\omega) = \frac{1}{1 - z_k\, ?^{-i\omega}}$$

If $|z_k| < 1$, we can write

$$X(\omega) = \sum_{n=0}^{\infty} (z_k)^n e^{-i\omega n}$$

$\Rightarrow x[n]$ is causal

So $\hat{r}[n]$ is the causal part of $\hat{q}[n]$:

$$\hat{r}[n] = \begin{cases} \tfrac{1}{2}\,\hat{q}[0] & ; \quad n = 0 \\ \hat{q}[n] & ; \quad n > 0 \\ 0 & ; \quad n < 0 \end{cases}$$

Algorithm:

Given the coefficients q[n] of the polynomial Q(z):

i. **Compute the M-point DFT of q[n] for a sufficiently large M.**

$$Q[k] = \sum_{n} q[n] e^{-i\frac{2\pi kn}{M}} \quad ; \quad 0 \leq k < M$$

ii. **Take the logarithm.**

$$\hat{Q}[k] = \ln\left(Q[k]\right)$$

iii. **Determine the complex cepstrum of q[n] by computing the IDFT.**

$$\hat{q}[n] = \frac{1}{M} \sum_{k=0}^{M-1} \hat{Q}[k]\, e^{i\frac{2\pi nk}{M}}$$

iv. **Find the causal part of $\hat{q}[n]$.**

$$\hat{r}[n] = \begin{cases} \frac{1}{2}\,\hat{q}[0] & ; \quad n = 0 \\ \hat{q}[n] & ; \quad n > 0 \\ 0 & ; \quad n < 0 \end{cases}$$

v. **Determine the DFT of r[n] by computing the exponent of the DFT of $\hat{r}[n]$.**

$$R[k] = \exp(\hat{R}[k]) = \exp\left(\sum_{k=0}^{M-1} \hat{r}[n] e^{-i\frac{2\pi}{M}kn}\right);\ 0 \leq k < M$$

vi. Determine the DFT of $h_0[n]$, by including half the zeros at z = -1.

$$H_0[k] = R[k]\,(1 + e^{-i\frac{2\pi k}{M}})^p$$

vii. Compute the IDFT to get $h_0[n]$.

$$h_0[n] = \frac{1}{M}\sum_{k=0}^{M-1} H_0[k]\, e^{i\frac{2\pi}{M}nk}$$