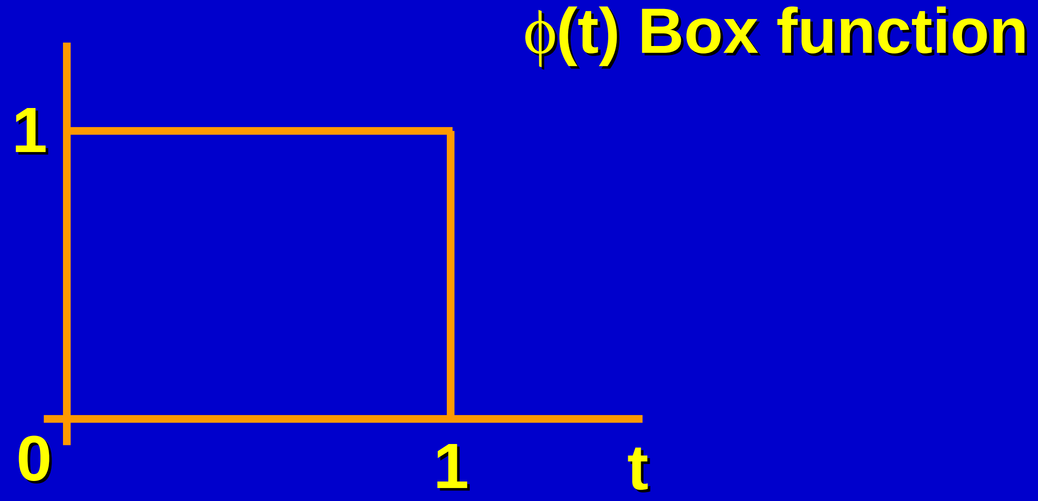
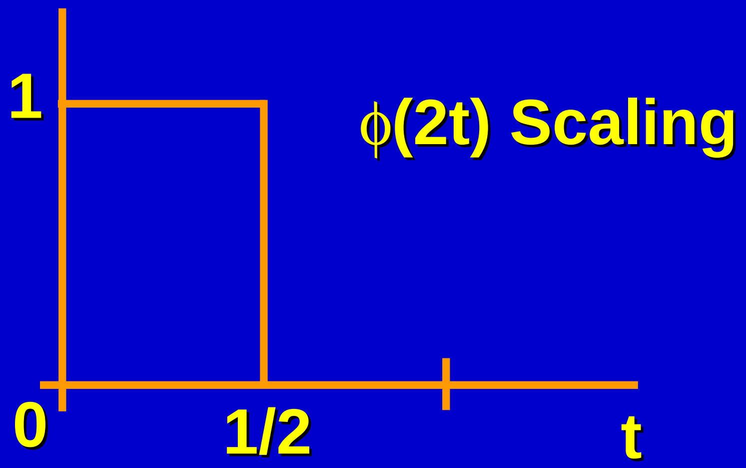
Course 18.327 and 1.130
Wavelets and Filter Banks

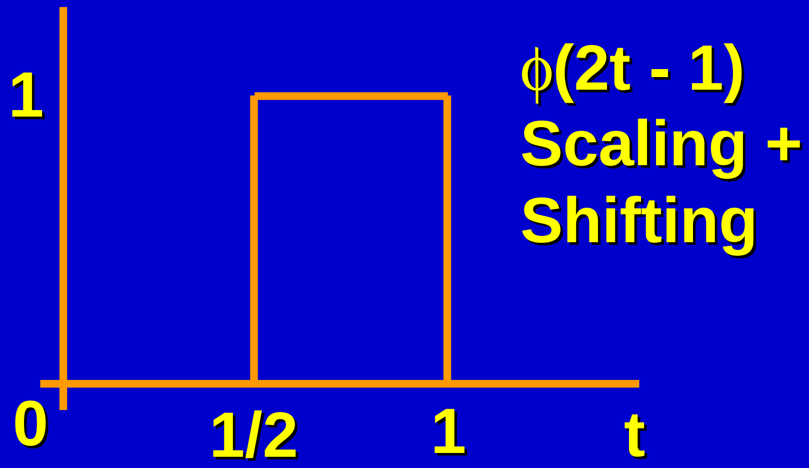
Multiresolution Analysis (MRA): Requirements for MRA; Nested Spaces and Complementary Spaces; Scaling Functions and Wavelets

Scaling Functions and Wavelets

Continuous time:







For this example:

$$\phi(t) = \phi(2t) + \phi(2t-1)$$

More generally:

$$\boxed{\phi(t) = 2\sum_{k=0}^{N} h_0[k]\phi(2t-k)}$$

Refinement equation
or
Two-scale difference equation

$\phi(t)$ is called a scaling function

The refinement equation couples the representations of a continuous-time function at two time scales. The continuous-time function is determined by a discrete-time filter, $h_0[n]$! For the above (Haar) example:

$$h_0[0] = h_0[1] = \tfrac{1}{2} \quad \text{(a lowpass filter)}$$

Note: (i) Solution to refinement equation may not always exist. If it does…

(ii) $\phi(t)$ has compact support i.e.

$\phi(t) = 0$ outside $0 \le t < N$

(comes from the FIR filter, $h_0[n]$)

(iii) $\phi(t)$ often has no closed form solution.

(iv) $\phi(t)$ is unlikely to be smooth.

Constraint on $h_0[n]$:

$$\int \phi(t)dt = 2 \sum_{k=0}^{N} h_0[k] \int \phi(2t - k)dt$$

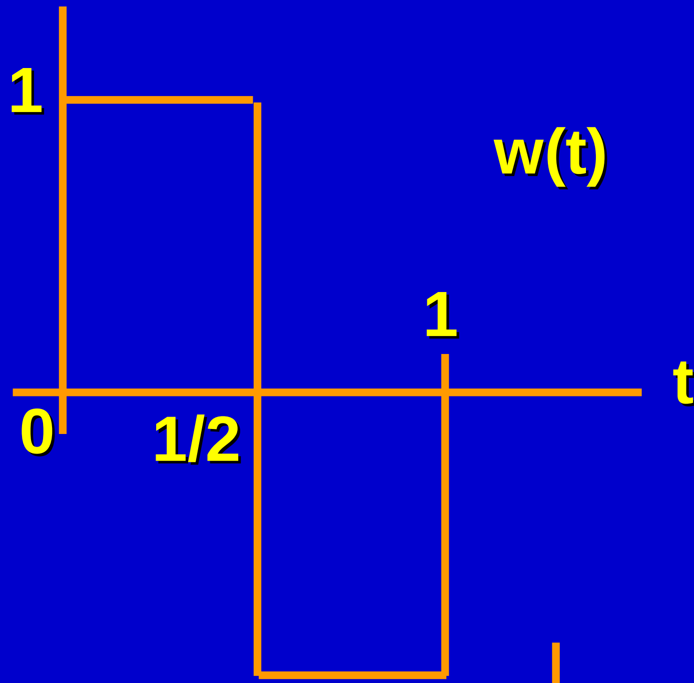
$$= 2 \sum_{k=0}^{N} h_0[k] \bullet \tfrac{1}{2} \int \phi(\tau)d\tau$$

So

$$\boxed{\sum_{k=0}^{N} h_0[k] = 1}$$

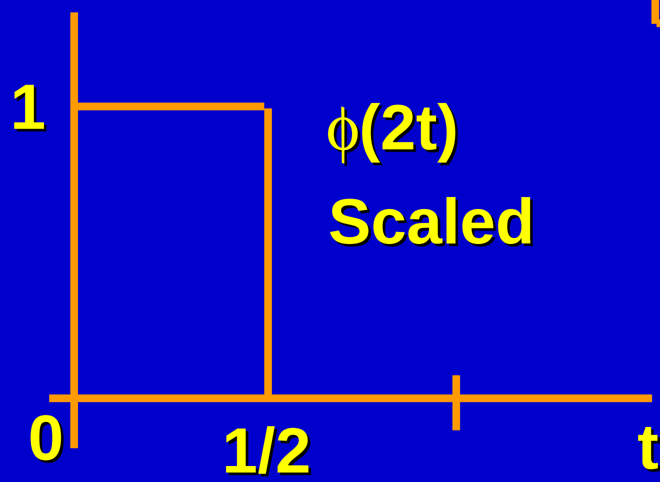
Assumes $\int \phi(t)dt \neq 0$

Now consider:



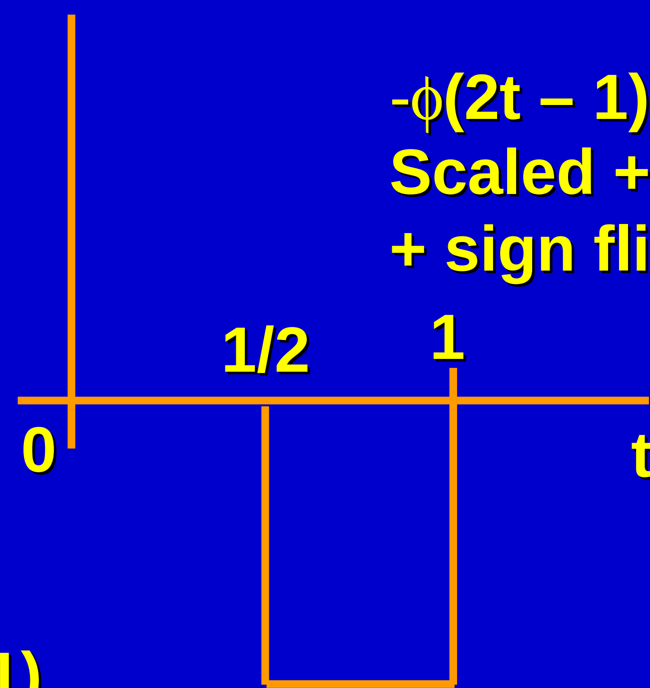
w(t)

Square wave
of finite length -
Haar wavelet



$\phi(2t)$

Scaled



$-\phi(2t - 1)$

Scaled + shifted
+ sign flipped

$$w(t) = \phi(2t) - \phi(2t - 1)$$

More generally:

$$\boxed{w(t) = 2\sum_{k=0}^{N} h_1[k]\, \phi(2t - k)}$$

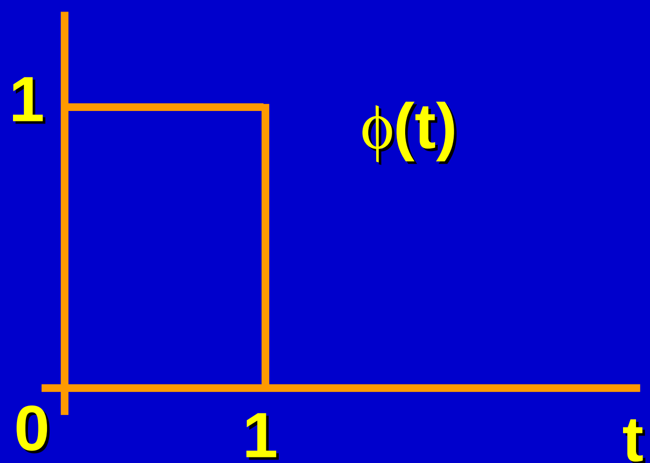
Wavelet equation

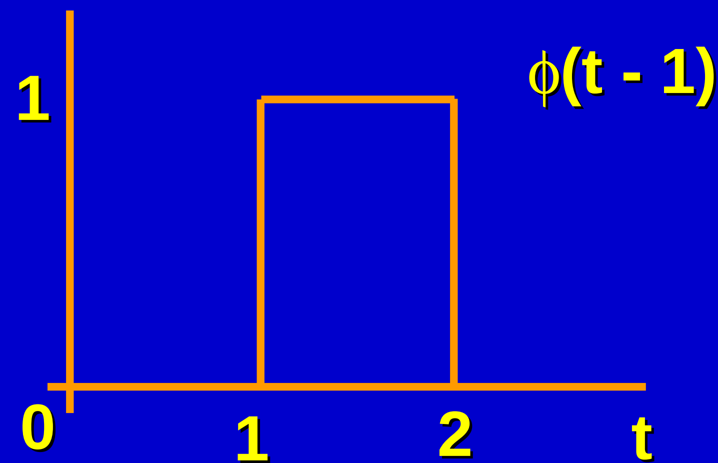
For the Haar wavelet example:

$h_1[0] = ½ \quad h_1[1] = -½$ **(a highpass filter)**

Some observations for Haar scaling function and wavelet

1. Orthogonality of integer shifts (translates):





$$\int \phi(t)\,\phi(t-k)dt = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

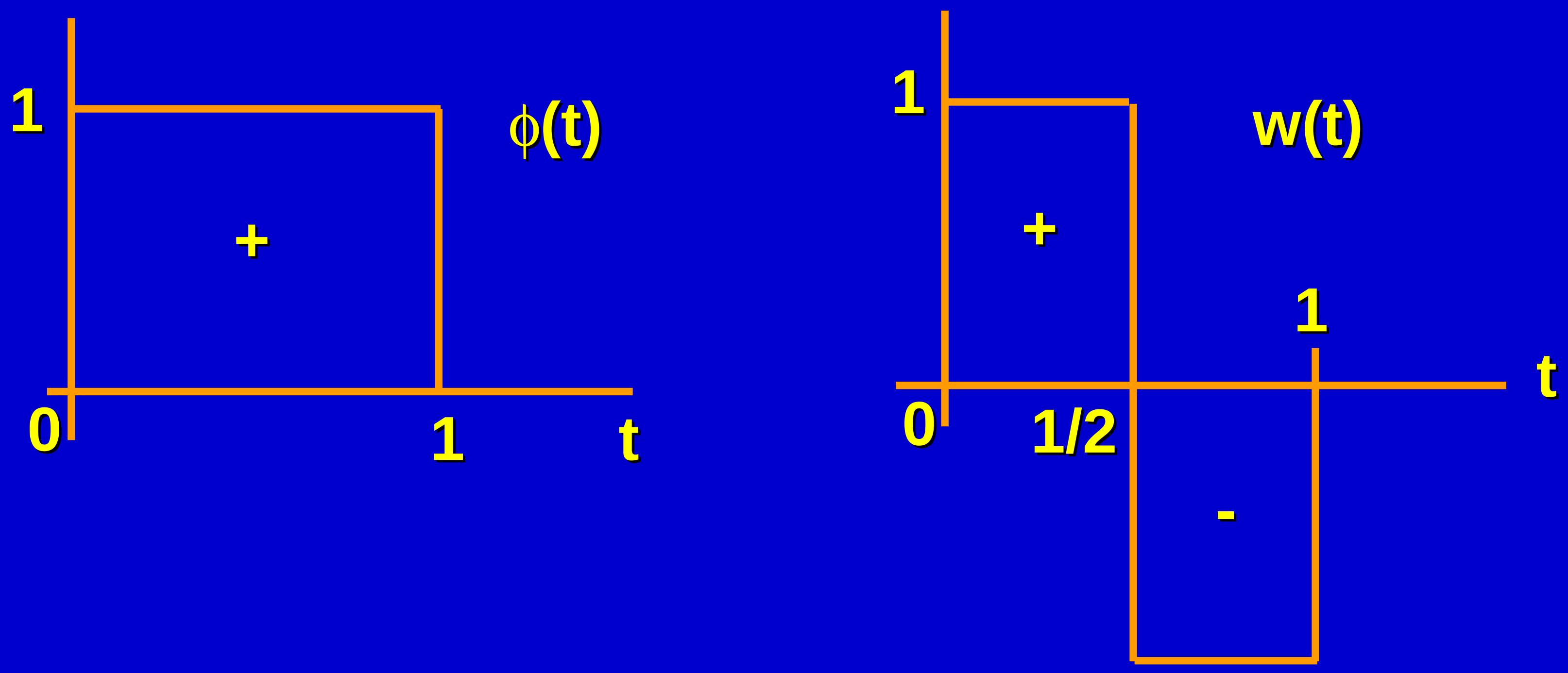
$$= \delta[k]$$

Similarly

$$\int w(t)\,w(t-k)dt = \delta[k]$$

Reason: no overlap

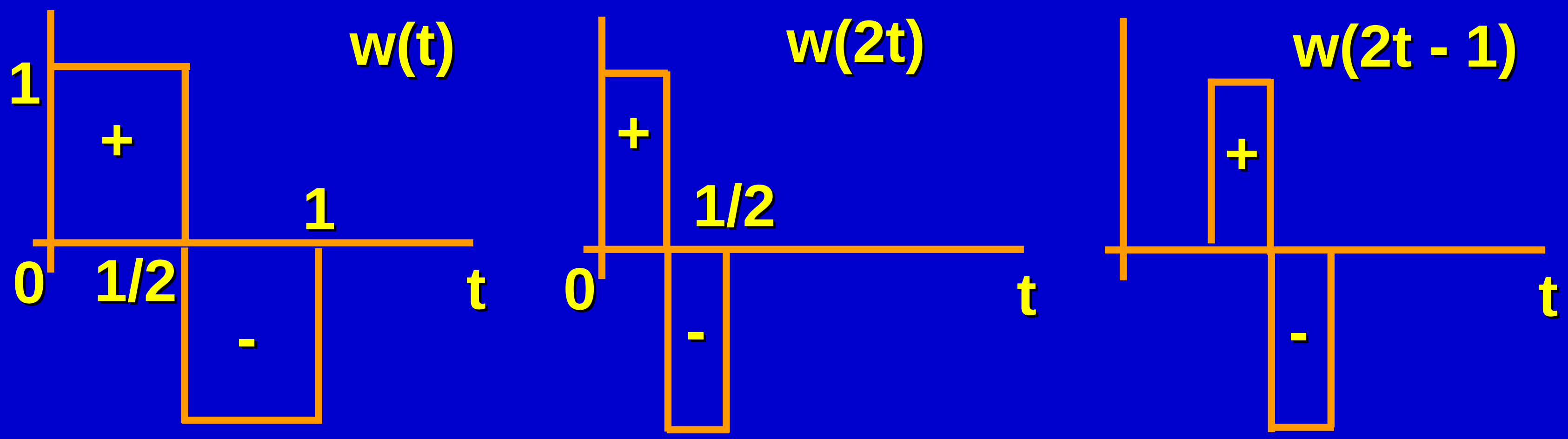
2. Scaling function is orthogonal to wavelet:



$\int \phi(t)\, w(t) dt = 0$

Reason: +ve and –ve areas cancel each other.

3. Wavelet is orthogonal across scales:



$$\int w(t)\, w(2t)dt = 0\,, \quad \int w(t)\, w(2t-1)dt = 0$$

Reason: finer scale versions change sign while coarse scale version remains constant.

Wavelet Bases

Our goal is to use w(t), its scaled versions (dilations) and their shifts (translates) as building blocks for continuous-time functions, f(t). Specifically, we are interested in the class of functions for which we can define the inner product:

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t)\, g^*(t)\,dt < \infty$$

Such functions f(t) must have finite energy:

$$\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2\, dt < \infty$$

and they are said to belong to the Hilbert space, $L^2(\Re)$.

Consider all dilations and translates of the Haar wavelet:

$$w_{j,k}(t) = 2^{j/2}\, w(2^j t - k) \quad ; \quad -\infty \le j \le \infty$$

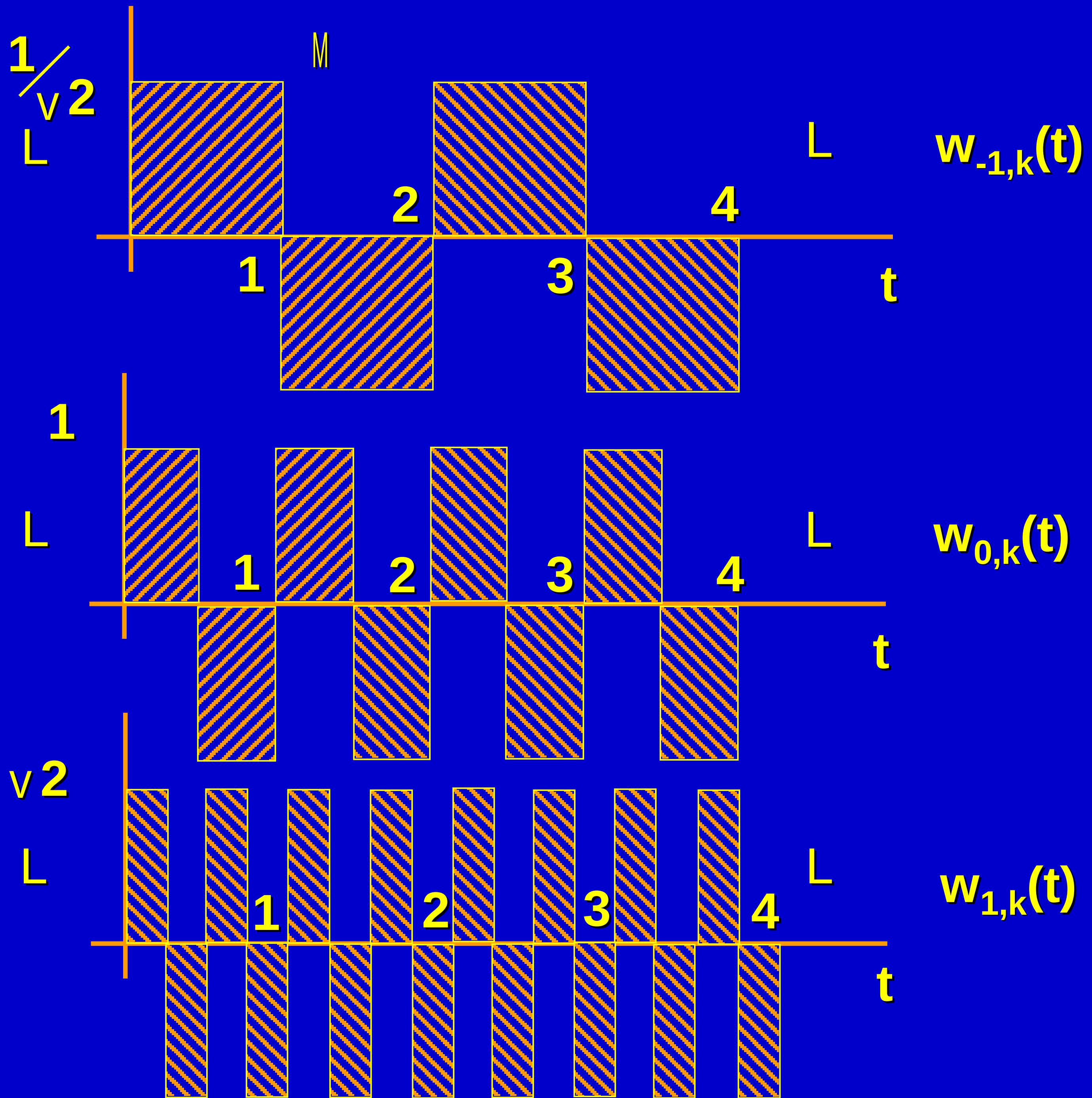
$$-\infty \le k \le \infty$$

Normalization factor so that $\|w_{j,k}(t)\| = 1$

$$\int w_{j,k}(t)\, w_{J,K}(t)\, dt = \int 2^{j/2}\, w(2^j t - k) \,.\, 2^{J/2}\, w(2^J t - K)dt$$

$$= \begin{cases} 1 \text{ if } j = J \text{ and } k = K \\ 0 \text{ otherwise} \end{cases}$$

$$= \delta[\, j - J\,]\, \delta[\, k - K\,]$$


1/√2
L
M
1
2
3
4
L
t
$w_{-1,k}(t)$
1
L
1
2
3
4
L
t
$w_{0,k}(t)$
√2
L
1
2
3
4
L
t
$w_{1,k}(t)$

$w_{jk}(t)$ form an orthonormal basis for $L^2(\Re)$.

$$f(t) = \sum_{j,k} b_{jk}\, w_{jk}(t) \quad ; \qquad w_{jk}(t) = 2^{j/2}\, w(2^j t - k)$$

$$b_{jk} = \int_{-\infty}^{\infty} f(t)\, w_{jk}(t)\, dt$$

Multiresolution Analysis

Key ingredients:

1. **A sequence of embedded subspaces:**

$$\{0\} \subset \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots \subset L^2(\Re)$$

$L^2(\Re)$ = all functions with finite energy

$$= \{f(t): \int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty\}$$

Hilbert space

Requirements:

- Completeness as $j \to \infty$. If $f(t)$ belongs to $L^2(\Re)$ and $f_j(t)$ is the portion of $f(t)$ that lies in V_j, then $\lim_{j \to \infty} f_j(t) = f(t)$

Restated as a condition on the subspaces:

$$\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(\Re)$$

- Emptiness as $j \rightarrow -\infty$

$$\lim_{j \rightarrow -\infty} \| f_j(t) \| = 0$$

Restated as a condition on the subspaces:

$$\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$$

2. A sequence of complementary subspaces, W_j, such that $V_j + W_j = V_{j+1}$

and $V_j \cap W_j = \{0\}$ (no overlap)

This is written as

$$V_j \oplus W_j = V_{j+1} \quad \text{(Direct sum)}$$

Note: An orthogonal multiresolution will have W_j orthogonal to V_j : $W_j \underline{\;?\;} V_j$.

So orthogonality will ensure that $V_j \cap W_j = \{0\}$

We thus have

$$
\begin{aligned}
V_1 &= V_0 \oplus W_0 \\
V_2 &= V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1 \\
V_3 &= V_2 \oplus W_2 = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \\
&\vdots \\
V_J &= V_{J-1} \oplus W_{J-1} = V_0 \oplus \sum_{j=0}^{J-1} W_j \\
&\vdots \\
L^2(\Re) &= V_0 \oplus \sum_{j=0}^{\infty} W_j
\end{aligned}
$$

We can also write the recursion for $j < 0$

$$
\begin{aligned}
V_0 &= V_{-1} \oplus W_{-1} \\
&= V_{-2} \oplus W_{-2} \oplus W_{-1} \\
&\quad \vdots \\
&= V_{-k} \oplus \sum_{j=-k}^{-1} W_j \\
&\quad \vdots \\
&= \sum_{j=-\infty}^{-1} W_j \qquad \Rightarrow L^2(\Re) = \sum_{j=-\infty}^{\infty} W_j
\end{aligned}
$$

3. A scaling (dilation) law:

If $f(t) \in V_j$ then $f(2t) \in V_{j+1}$

4. A shift (translation) law:

If $f(t) \in V_j$ then $f(t-k) \in V_j$ k integer

5. V_0 has a shift-invariant basis, $\{\phi(t-k) : -\infty \le k \le \infty\}$

W_0 has a shift-invariant basis, $\{w(t-k) : -\infty \le k \le \infty\}$

We expect that $V_1 = V_0 + W_0$ will have twice as many basis functions as V_0 alone.

First possibility: $\{\phi(t-k), w(t-k) : -\infty \le k \le \infty\}$

Second possibility: use the scaling law i.e.

if $\phi(t-k) \in V_0$, then $\phi(2t-k) \in V_1$

So

V_1 has a shift-invariant basis, $\{\sqrt{2}\,\phi(2t-k): -\infty \le k \le \infty\}$

Can we relate this basis for V_1 to the basis for V_0?

We know that

$$V_0 \subset V_1$$

So any function in V_0 can be written as a combination of the basic functions for V_1.

In particular, since $\phi(t) \in V_0$, we can write

$$\boxed{\phi(t) = 2\sum_k h_0[k]\,\phi(2t-k)}$$

This is the Refinement Equation (a.k.a. the Two-Scale Difference Equation or the Dilation Equation).

We also know that

$$W_0 = V_1 - V_0$$

So

$$W_0 \subset V_1$$

This means that any function in W_0 can also be written as a combination of the basic functions for V_1.

Since $w(t) \in W_0$, we can write

$$\boxed{w(t) = 2\sum_{k} h_1[k]\, \phi(2t - k)}$$

Wavelet Equation

Multiresolution Representations

Functions:

$$L^2(\Re) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots$$

Level 2 detail

Level 1 detail

Level 0 detail

Finite energy functions

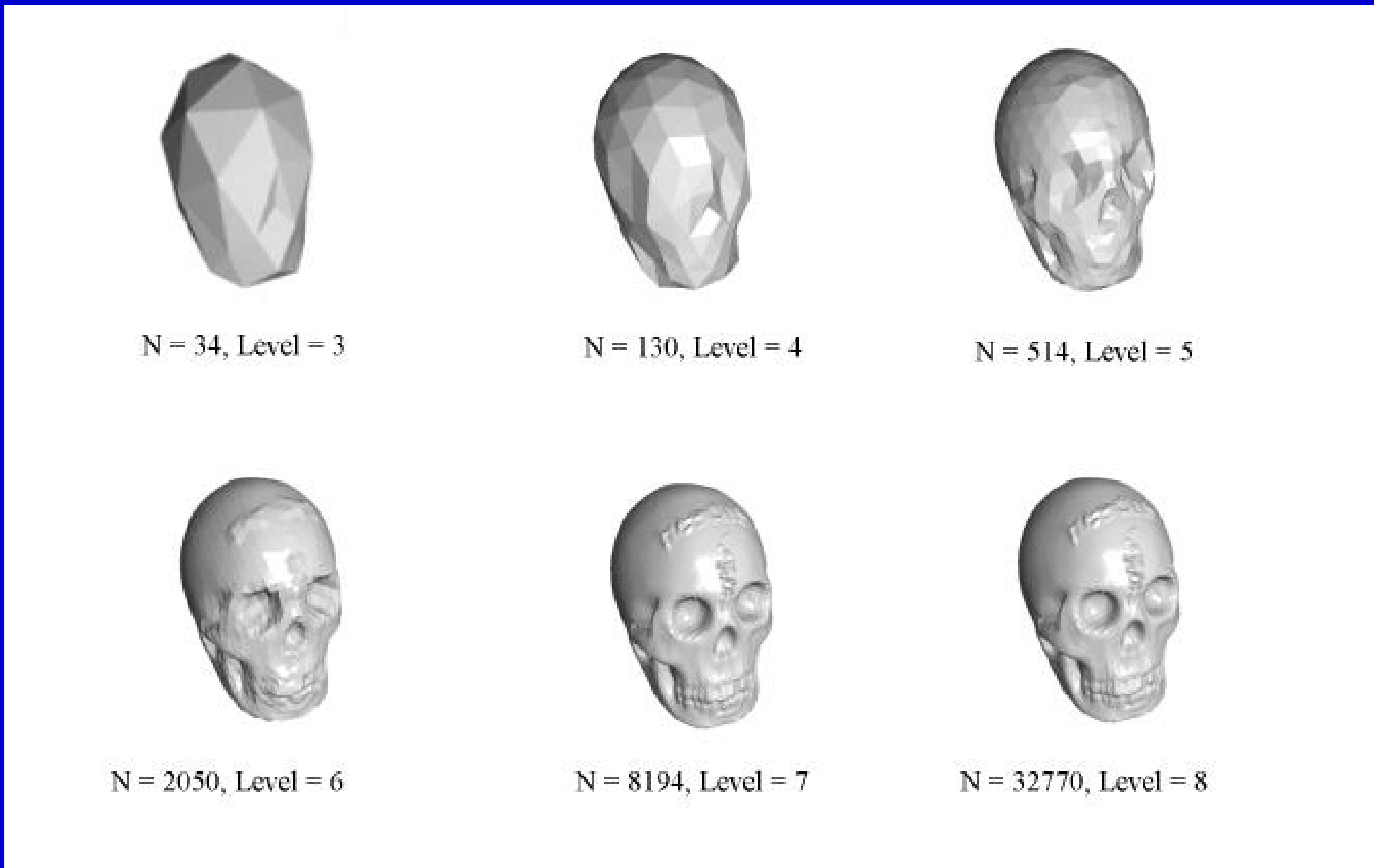
Coarse approximation

Images:

V_0 W_0 V_1 W_1 V_2

Multiresolution Representations

Geometry:



Mesh courtesy of Igor Guskov (Caltech)

Course 18.327 and 1.130
Wavelets and Filter Banks

Refinement Equation: Iterative and Recursive Solution Techniques; Infinite Product Formula; Filter Bank Approach for Computing Scaling Functions and Wavelets

Solution of the Refinement Equation

$$\phi(t) = 2\sum_{k=0}^{N} h_0[k]\, \phi(2t-k)$$

First, note that the solution to this equation may not always exist! The existence of the solution will depend on the discrete-time filter $h_0[k]$.

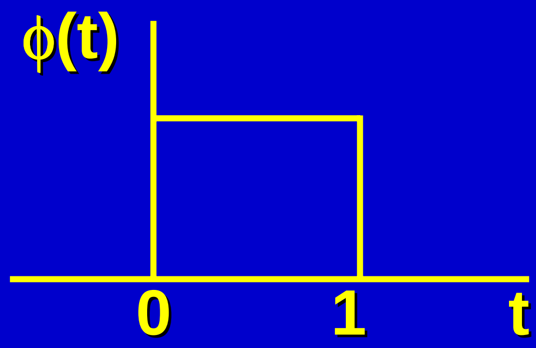
If the solution does exist, it is unlikely that $\phi(t)$ will have a closed form solution. The solution is also unlikely to be smooth. We will see, however, that if $h_0[n]$ is FIR with

$$h_0[n] = 0 \quad \text{outside} \quad 0 \leq n \leq N$$

then $\phi(t)$ has compact support:

$$\phi(t) = 0 \quad \text{outside} \quad 0 < t < N$$

Approach 1 Iterate the box function



$\phi^{(0)}(t)$ = box function on [0 , 1]

$$\phi^{(i+1)}(t) = 2\sum_{k=0}^{N} h_0[k]\, \phi^{(i)}(2t-k)$$

If the iteration converges, the solution will be given by

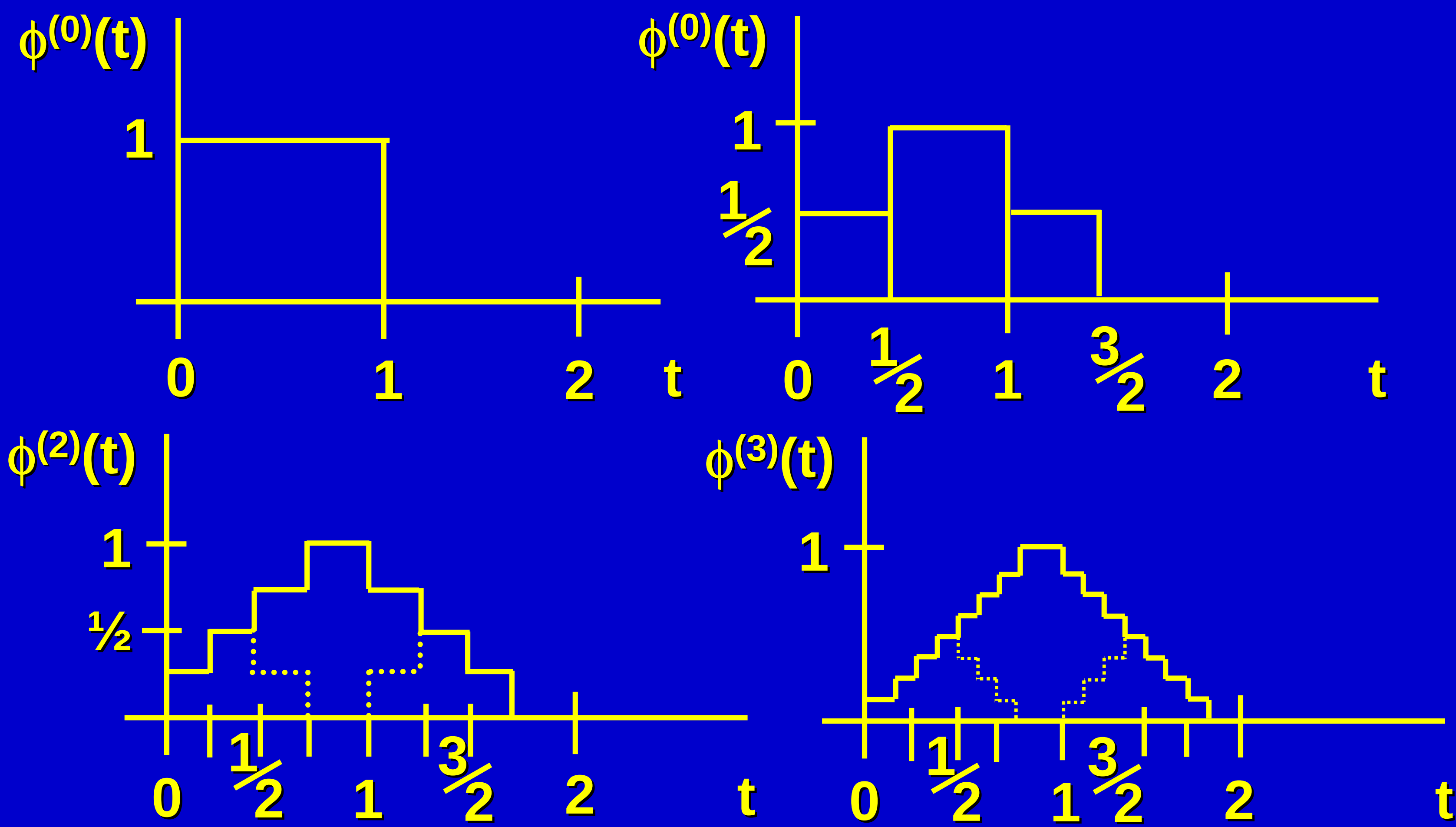
$$\lim_{i \to \infty} \phi^{(i)}(t)$$

This is known as the **cascade algorithm**.

Example: suppose $h_0[k] = \{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$

$$\phi^{(i+1)}(t) = \tfrac{1}{2}\,\phi^{(i)}(2t) + \phi^{(i)}(2t-1) + \tfrac{1}{2}\,\phi^{(i)}(2t-2)$$

Then



Converges to the hat function on [0, 2]

Approach 2   Use recursion

First solve for the values of $\phi(t)$ at integer values of t.

Then solve for $\phi(t)$ at half integer values, then at quarter integer values and so on.

This gives us a set of discrete values of the scaling function at all dyadic points $t = n/2^i$.

At integer points:

$$\phi(n) = 2 \sum_{k=0}^{N} h_0[k]\, \phi(2n - k)$$

Suppose N = 3

$$\phi(0) = 2 \sum_{k=0}^{3} h_0[k]\, \phi(-k)$$

$$\phi(1) = 2 \sum_{k=0}^{3} h_0[k]\, \phi(2-k)$$

$$\phi(2) = 2 \sum_{k=0}^{3} h_0[k]\, \phi(4-k)$$

$$\phi(3) = 2 \sum_{k=0}^{3} h_0[k]\, \phi(6-k)$$

Using the fact that $\phi(n) = 0$ for $n < 0$ and $n > N$, we can write this in matrix form as

$$\begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix} = 2 \begin{bmatrix} h_0[0] & & & \\ h_0[2] & h_0[1] & h_0[0] & \\ & h_0[3] & h_0[2] & h_0[1] \\ & & & h_0[3] \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix}$$

Notice that this is an eigenvalue problem

$$\lambda\Phi = A\Phi$$

where the eigenvector is the vector of scaling function values at integer points and the eigenvalue is $\lambda = 1$.

Note about normalization:
Since $(A - \lambda I)\,\Phi = 0$ has a non-unique solution, we must choose an appropriate normalization for Φ
The correct normalization is

$$\sum_n \phi(n) = 1$$

This comes from the fact that we need to satisfy the partition of unity condition, $\sum_n \phi(x-n) = 1$.

At half integer points:

$$\phi(n/2) = 2 \sum_{k=0}^{N} h_0[k]\, \phi(n-k)$$

So, for N = 3, we have

$$\begin{bmatrix} \phi(1/2) \\ \phi(3/2) \\ \phi(5/2) \\ \ \end{bmatrix} = 2 \begin{bmatrix} h_0[1] & h_0[0] & & \\ h_0[3] & h_0[2] & h_0[1] & h_0[0] \\ & & h_0[3] & h_0[2] \\ & & & \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix}$$

Scaling Relation and Wavelet Equation in Frequency Domain

$$\phi(t) = 2 \sum_k h_0[k]\, \phi(2t - k)$$

$$\int_{-\infty}^{\infty} \phi(t) e^{-i\Omega t}\, dt = 2 \sum_k h_0[k] \int_{-\infty}^{\infty} \phi(2t - k)\, e^{-i\Omega t}\, dt$$

$$= 2 \sum_k h_0[k]\, \tfrac{1}{2} \int_{-\infty}^{\infty} \phi(\tau) e^{-i\Omega(\tau + k)/2}\, d\tau$$

$$= \sum_k h_0[k] e^{-i\Omega k/2} \int_{-\infty}^{\infty} \phi(\tau)\, e^{-i\Omega\tau/2}\, d\tau$$

$$\text{i.e. } \hat{\phi}(\Omega) = H_0\left(\frac{\Omega}{2}\right) \cdot \hat{\phi}\left(\frac{\Omega}{2}\right)$$

$$= H_0\left(\frac{\Omega}{2}\right) \cdot H_0\left(\frac{\Omega}{4}\right) \cdot \hat{\phi}\left(\frac{\Omega}{4}\right)$$

$$\vdots$$

$$= \left\{ \prod_{j=1}^{\infty} H_0\left(\frac{\Omega}{2^j}\right) \right\} \hat{\phi}(0)$$

$$\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(t)\, dt = 1 \text{ (Area is normalized to 1)}$$

So

$$\boxed{\hat{\phi}(\Omega) = \prod_{j=1}^{\infty} H_0\left(\frac{\Omega}{2^j}\right)}$$

Infinite Product Formula

Similarly

$$w(t) = 2 \sum_{k} h_1[k]\, \phi(2t - k)$$

leads to

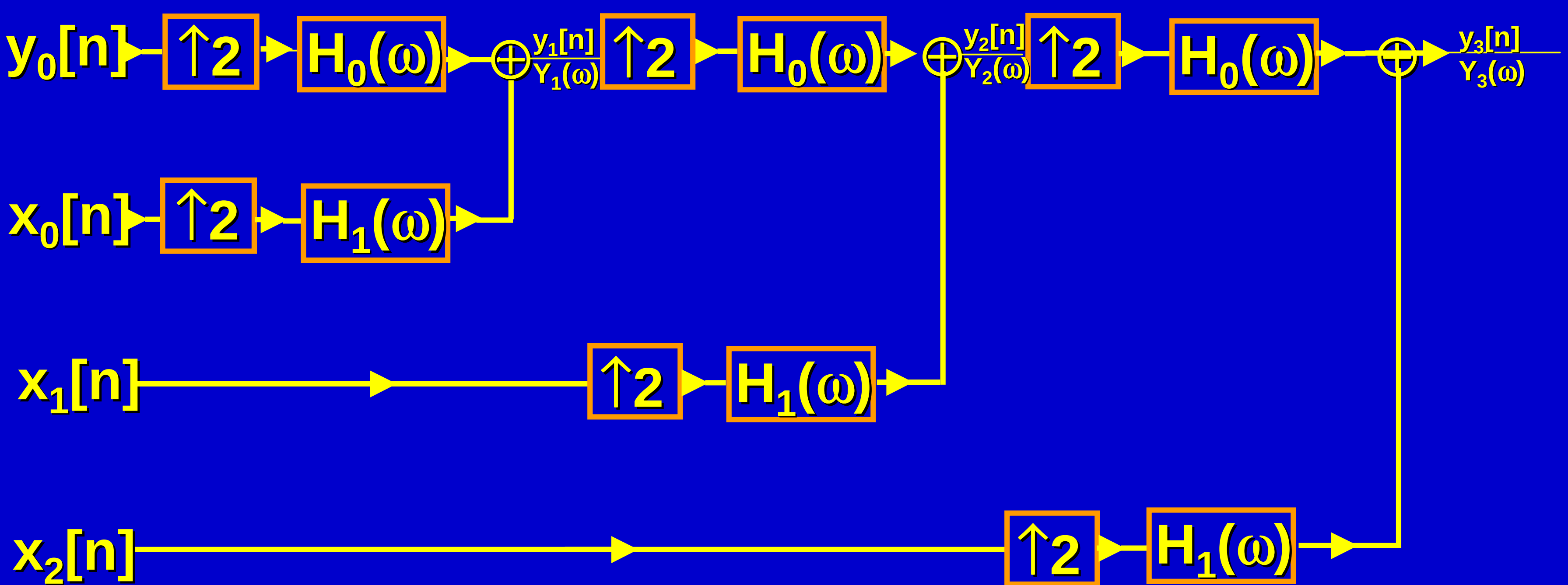
$$\boxed{\hat{w}(\Omega) = H_1\left(\frac{\Omega}{2}\right) \hat{\phi}\left(\frac{\Omega}{2}\right)}$$

Desirable properties for $H_0(\omega)$:

- $H(0) = 1$, so that $\hat{\phi}(0) = 1$
- $H(\omega)$ should decay to zero as $\omega \to \pi$,

so that $\int_{-\infty}^{\infty} |\hat{\phi}(\Omega)|^2 d\,\Omega < \infty$

Computation of the Scaling Function and Wavelet – Filter Bank Approach



Normalize so that $\sum_{n} h_0[n] = 1.$

i. Suppose $y_0[n] = \delta[n]$ and $x_k[n] = 0$.

$$Y_0(\omega) = 1$$
$$Y_1(\omega) = Y_0(2\omega)\,H_0(\omega) = H_0(\omega)$$
$$Y_2(\omega) = Y_1(2\omega)\,H_0(\omega) = H_0(2\omega)H_0(\omega)$$
$$Y_3(\omega) = Y_2(2\omega)\,H_0(\omega) = H_0(4\omega)\,H_0(2\omega)\,H_0(\omega)$$

After K iterations:

$$Y_K(\omega) = \prod_{k=0}^{K-1} H_0(2^k\omega)$$

What happens to the sampling period?

Sampling period at input $= T_0 = 1$ (say)

Sampling period at output $= T_K = \frac{1}{2}^K$

Treat the output as samples of a continuous time signal, $y_K^c(t)$, with sampling period $\frac{1}{2^K}$:

$$y_K[n] = \frac{1}{2^K} y_K^c(n/2^K)$$

$$\Rightarrow Y_K(\omega) = \hat{Y}_K^c(2^K\omega) \quad ; \quad -\pi \leq \omega \leq \pi$$

($y_K^c(t)$ is chosen to be bandlimited)

Replace $2^K\omega$ with Ω:

$$\hat{Y}_K^c(\Omega) = Y_K(\Omega/2^k) = \prod_{k=0}^{K-1} H_0(\Omega/2^{K-k}) = \prod_{j=1}^{K} H_0(\Omega/2^j) \ ;$$

$$-2^K\pi \leq \Omega \leq 2^K\pi$$

So

$$\lim_{k\to\infty} \hat{Y}_K^c(\Omega) = \prod_{j=1}^{\infty} H_0(\Omega/2^j) = \hat{\phi}(\Omega)$$

$\Rightarrow$ $\boxed{\text{$2^K y_K[n]$ converges to the samples of the scaling function, $\phi(t)$, taken at $t = n/2^K$.}}$

ii. Suppose $y_0[n] = 0$, $x_0[n] = \delta[n]$ and all other $x_k[n] = 0$

$$Y_K(\omega) = H_1(2^{K-1}\omega)\prod_{k=0}^{K-2} H_0(2^k\omega)$$

Then

$$\hat{Y}_K^c(\Omega) = Y_K(\Omega/2^K) = H_1\left(\frac{\Omega}{2}\right)\prod_{k=0}^{K-2} H_0(\Omega/2^{K-k})$$

$$= H_1\left(\frac{\Omega}{2}\right)\prod_{j=1}^{K-1} H_0\left(\frac{1}{2}\cdot\frac{\Omega}{2^j}\right)$$

So

$$\lim_{K\to\infty} \hat{Y}_K^c(\Omega) = H_1(\Omega/2)\,\hat{\phi}(\Omega/2) = \hat{w}(\Omega)$$

$\Rightarrow$ $\boxed{\text{$2^K y_K[n]$ converges to the samples of the wavelet, $w(t)$, taken at $t = n/2^K$.}}$

Support of the Scaling Function

length $\{v[n]\}$ = 2 • length $\{y_{k-1}[n]\}$ - 1

Suppose that

$$h_0[n] = 0 \text{ for } n < 0 \text{ and } n > N$$

$\Rightarrow$ length $\{y_k[n]\}$ = length $\{v[n]\}$ + length $\{h_0[n]\}$ – 1

= 2 • length $\{y_{K-1}[n]\}$ + N – 1

Solve the recursion with length $\{y_0[n]\}$ = 1

So

$$\text{length } \{y_k[n]\} = (2^K - 1)N + 1$$

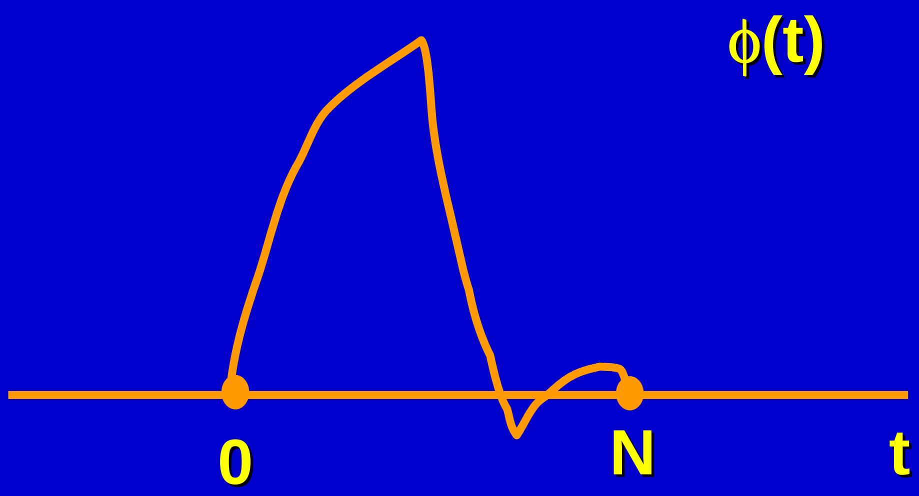
i.e. length $\{y_K^c(t)\}$ = T_K . length $\{y_K[n]\}$

$$= \frac{(2^K - 1)\,N + 1}{2^K}$$

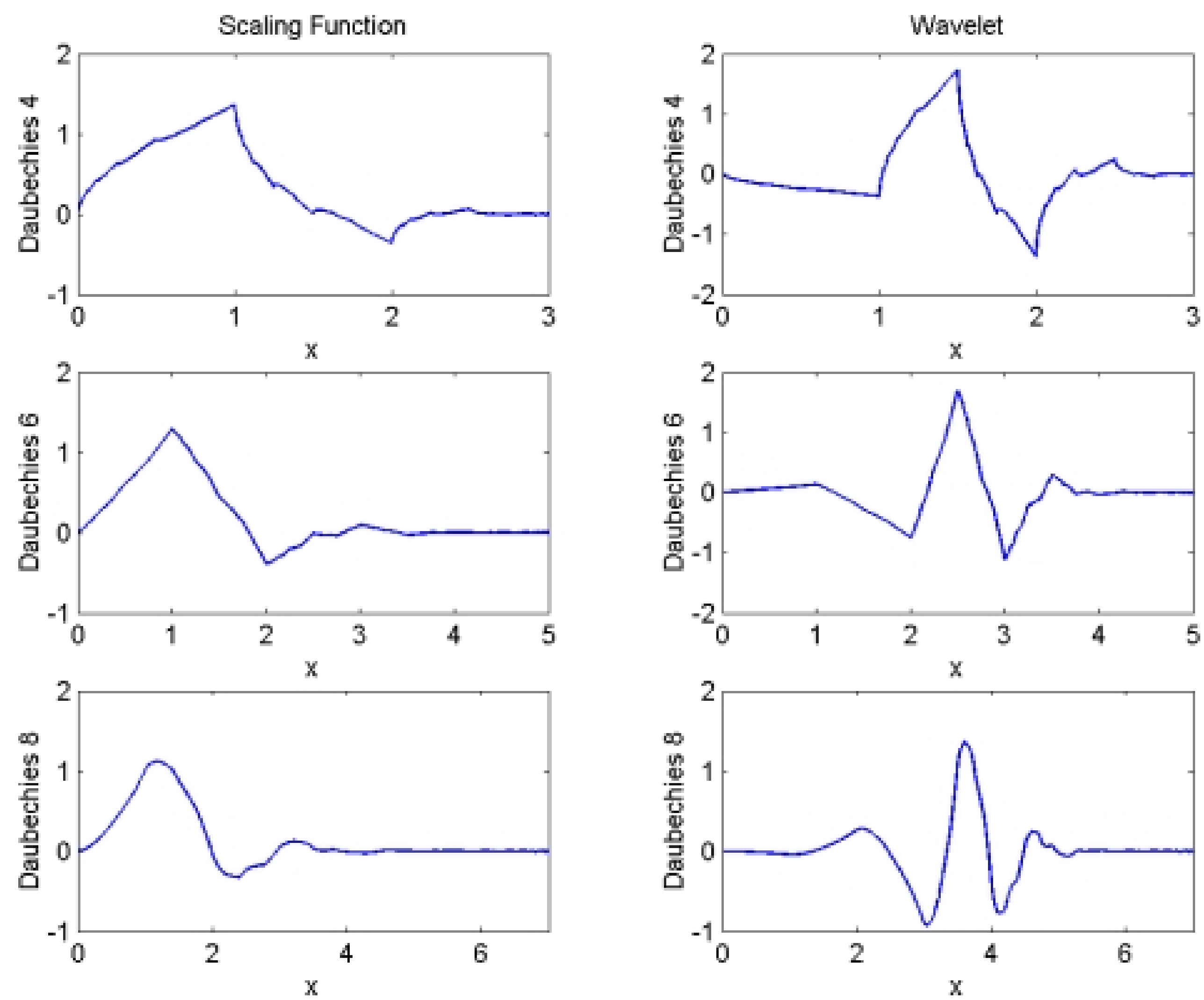
$$= N - \frac{N-1}{2^K}$$

$\lim_{K \to \infty}$

length $\{\phi(t)\}$ = N



So the scaling function is supported on the interval [0, N]

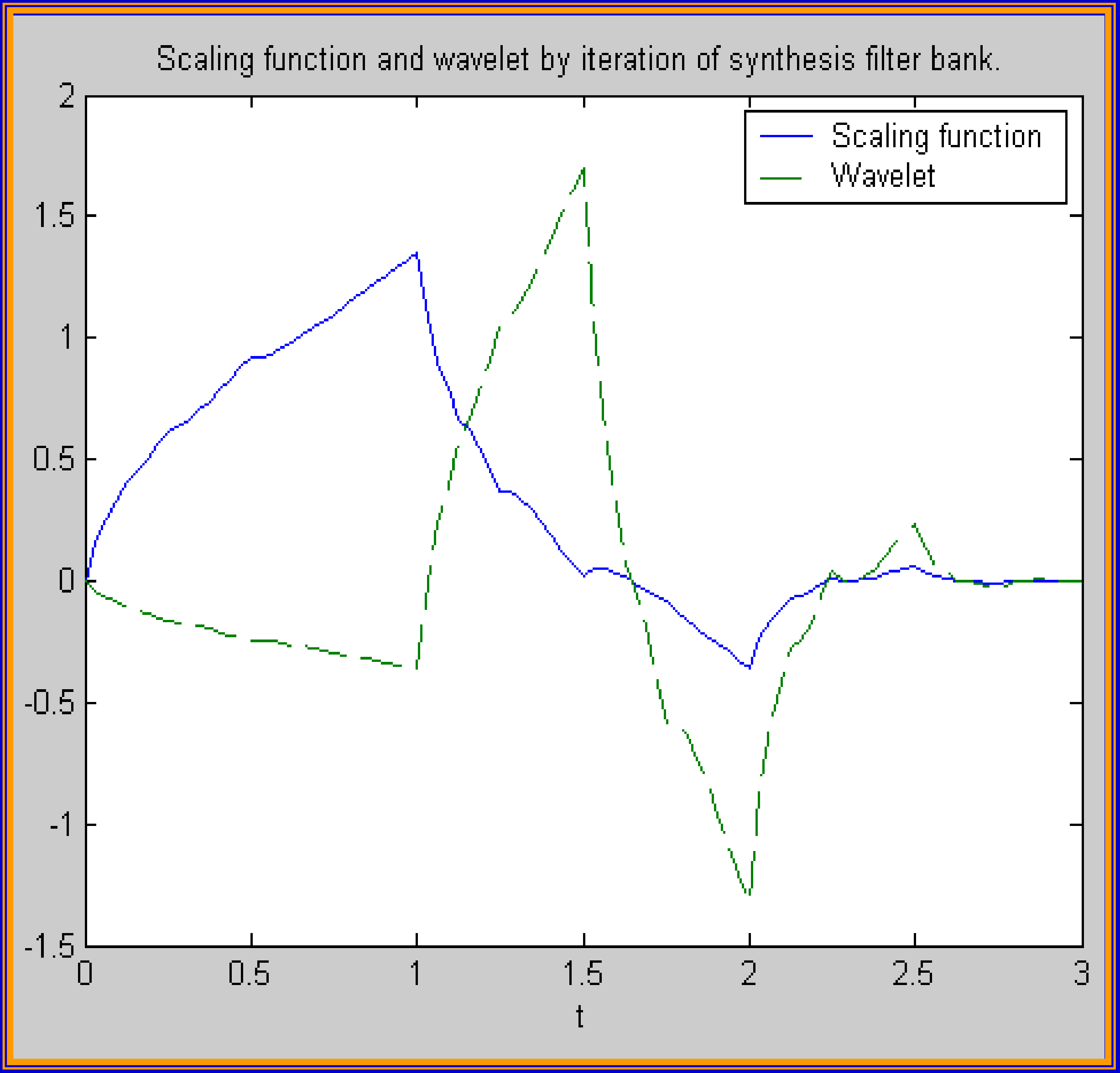

Scaling Function
Wavelet
Daubechies 4
Daubechies 6
Daubechies 8
x

Matlab Example 6

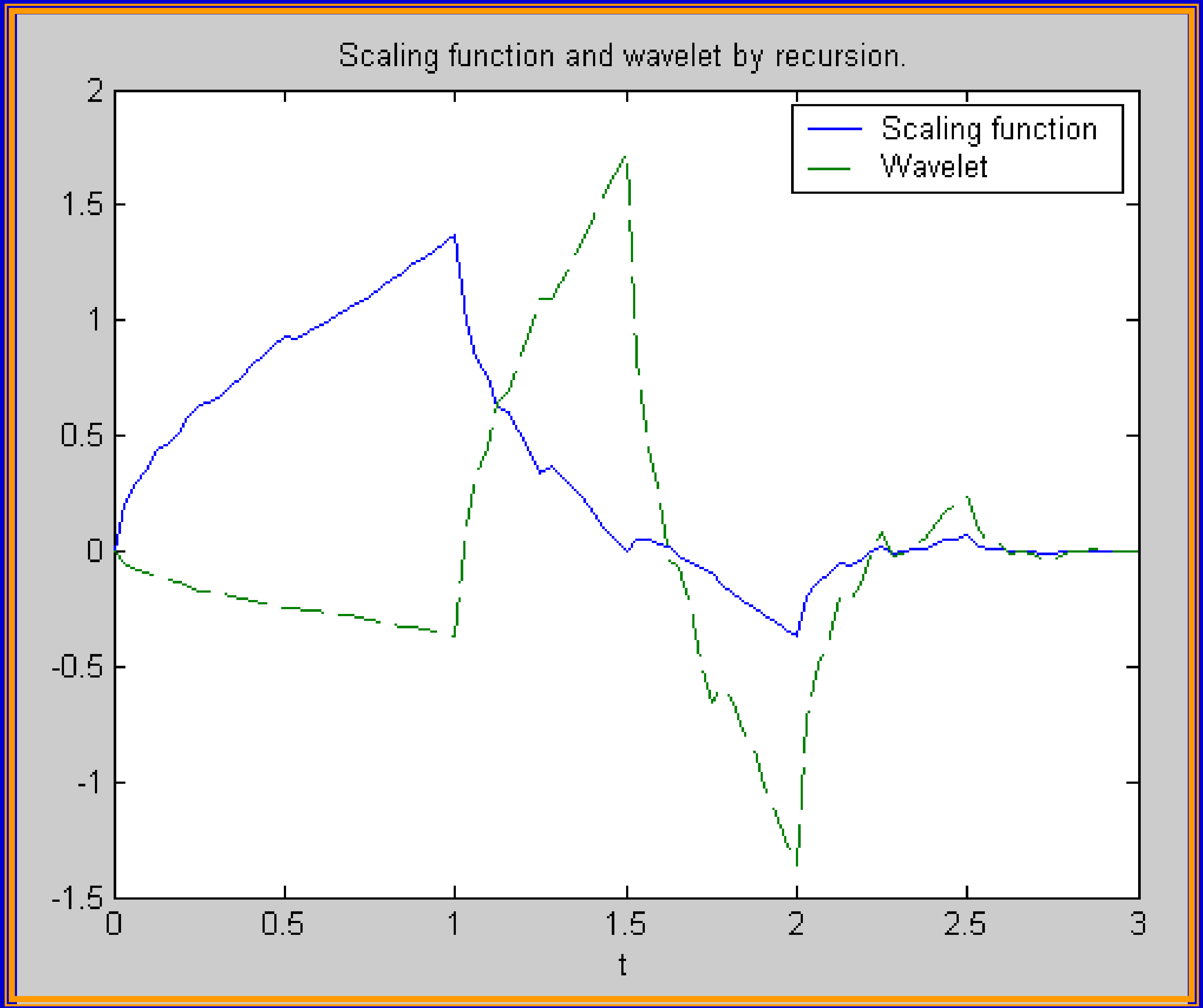
Generation of orthogonal scaling functions and wavelets

MATLAB M-file MATLAB M-file

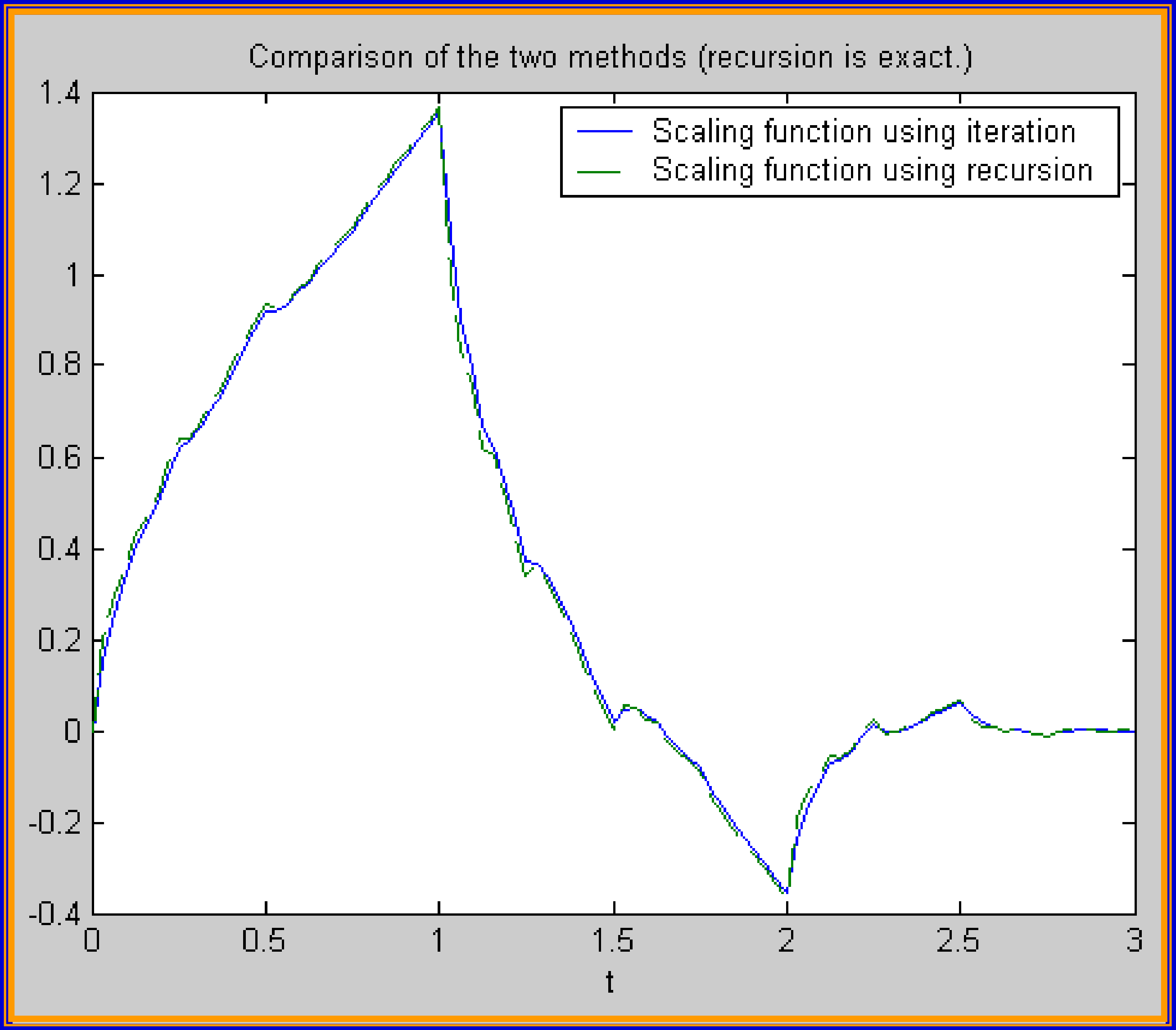
By Inverse DWT



By Recursion



Comparison

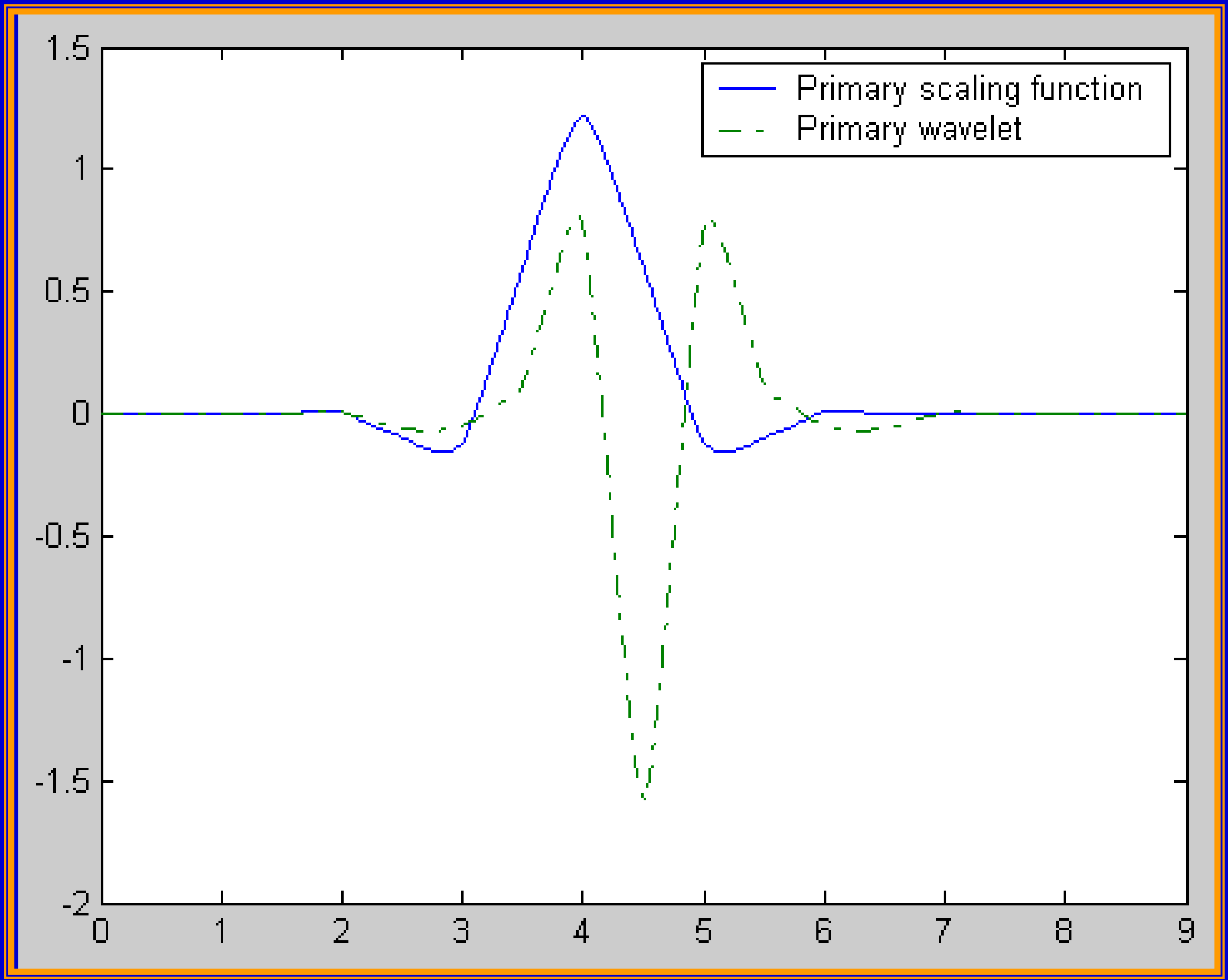

Comparison of the two methods (recursion is exact.)
Scaling function using iteration
Scaling function using recursion
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
0.5
1
1.5
2
2.5
3
t

Matlab Example 7

Generation of biorthogonal scaling functions and wavelets.

MATLAB M-file

Primary Daub 9/7 Pair



Dual Daub 9/7 Pair