

Course 18.327 and 1.130

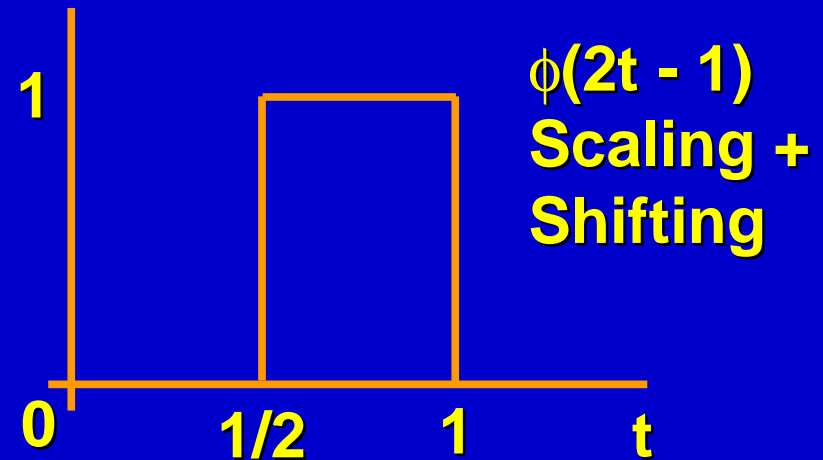
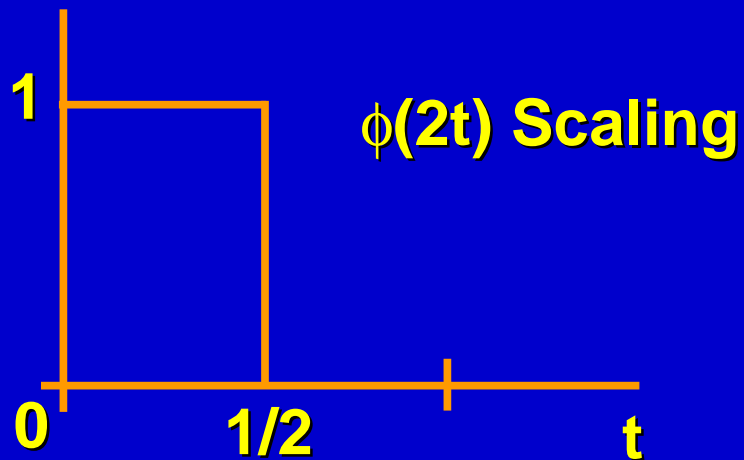
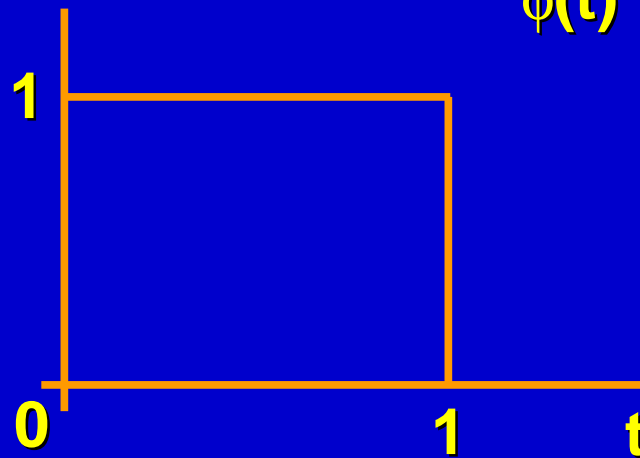
Wavelets and Filter Banks

Multiresolution Analysis (MRA):
Requirements for MRA;
Nested Spaces and
Complementary Spaces;
Scaling Functions and Wavelets

Scaling Functions and Wavelets

Continuous time:

$\phi(t)$ Box function



For this example:

$$\phi(t) = \phi(2t) + \phi(2t - 1)$$

More generally:

$$\phi(t) = 2 \sum_{k=0}^N h_0[k] \phi(2t - k)$$

**Refinement equation
or
Two-scale difference
equation**

$\phi(t)$ is called a scaling function

The refinement equation couples the representations of a continuous-time function at two time scales. The continuous-time function is determined by a discrete-time filter, $h_0[n]$! For the above (Haar) example:

$$h_0[0] = h_0[1] = \frac{1}{2} \quad (\text{a lowpass filter})$$

Note: (i) Solution to refinement equation may not always exist. If it does...

(ii) $\phi(t)$ has compact support i.e.

$$\phi(t) = 0 \text{ outside } 0 \leq t < N$$

(comes from the FIR filter, $h_0[n]$)

(iii) $\phi(t)$ often has no closed form solution.

(iv) $\phi(t)$ is unlikely to be smooth.

Constraint on $h_0[n]$:

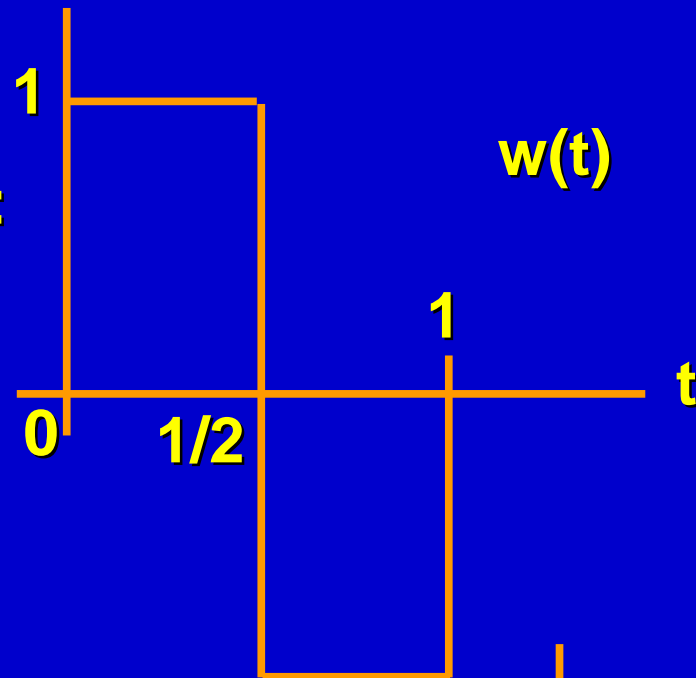
$$\begin{aligned} \int \phi(t) dt &= 2 \sum_{k=0}^N h_0[k] \int \phi(2t - k) dt \\ &= 2 \sum_{k=0}^N h_0[k] \cdot \frac{1}{2} \int \phi(\tau) d\tau \end{aligned}$$

So

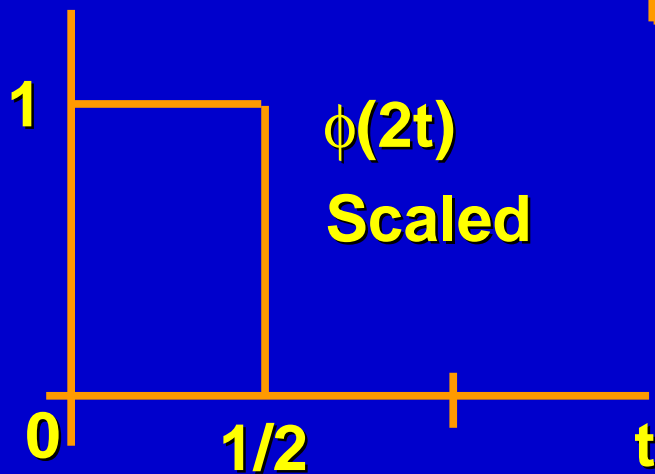
$$\sum_{k=0}^N h_0[k] = 1$$

Assumes $\int \phi(t) dt \neq 0$

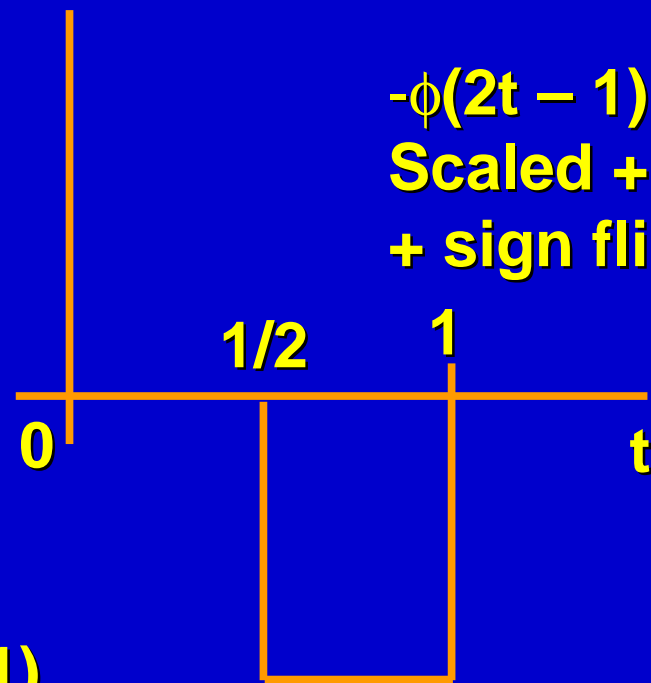
Now consider:



Square wave
of finite length -
Haar wavelet



$\phi(2t)$
Scaled



$-\phi(2t-1)$
Scaled + shifted
+ sign flipped

$$w(t) = \phi(2t) - \phi(2t - 1)$$

More generally:

$$w(t) = 2 \sum_{k=0}^N h_1[k] \phi(2t - k)$$

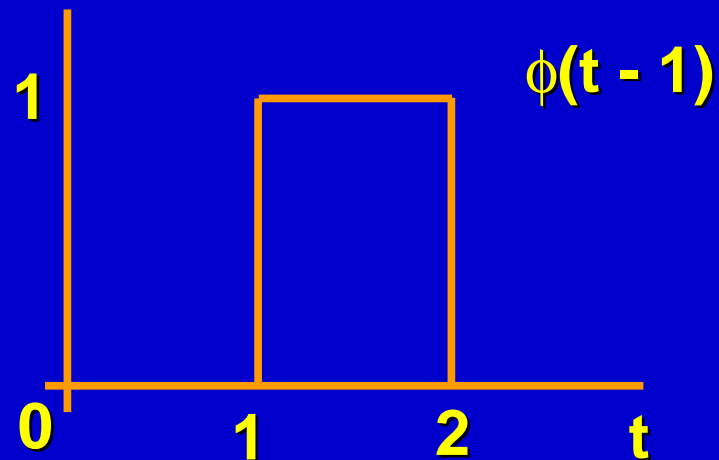
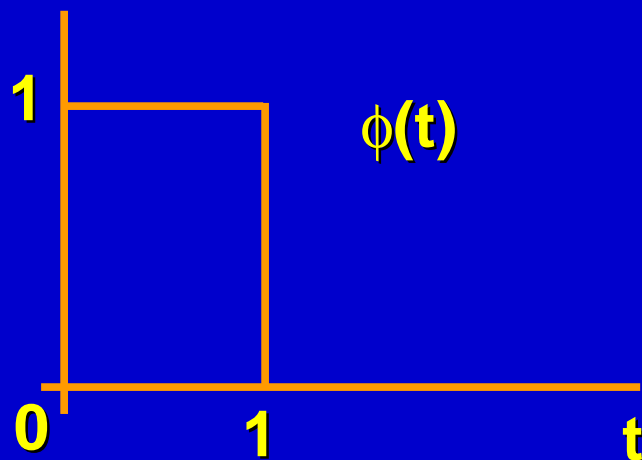
Wavelet equation

For the Haar wavelet example:

$$h_1[0] = 1/2 \quad h_1[1] = -1/2 \quad \text{(a highpass filter)}$$

Some observations for Haar scaling function and wavelet

1. Orthogonality of integer shifts (translates):



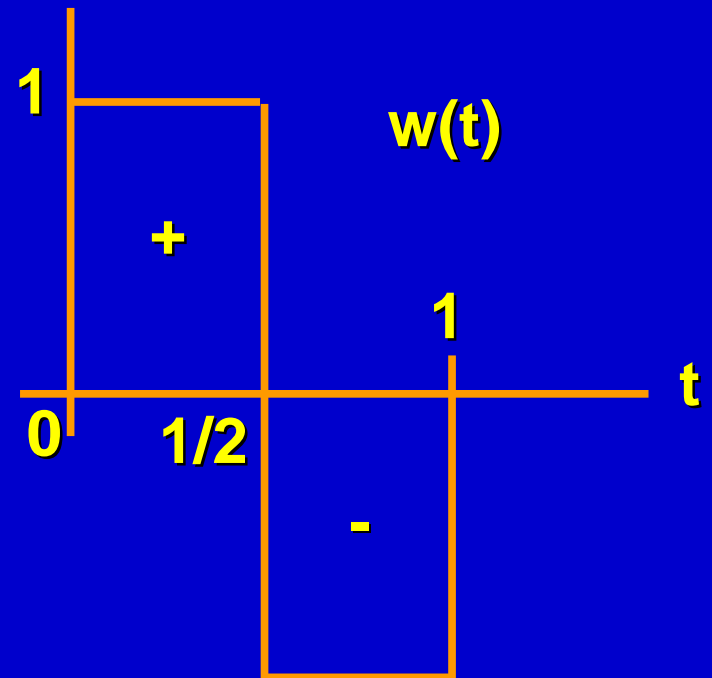
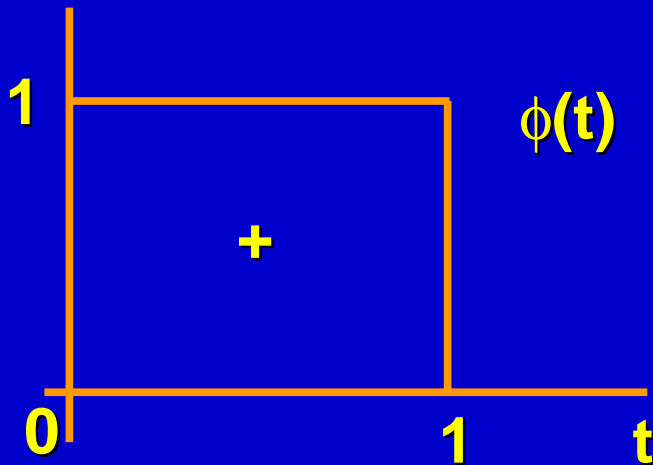
$$\int \phi(t) \phi(t-k) dt = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$
$$= \delta[k]$$

Similarly

$$\int w(t) w(t-k) dt = \delta[k]$$

Reason: no overlap

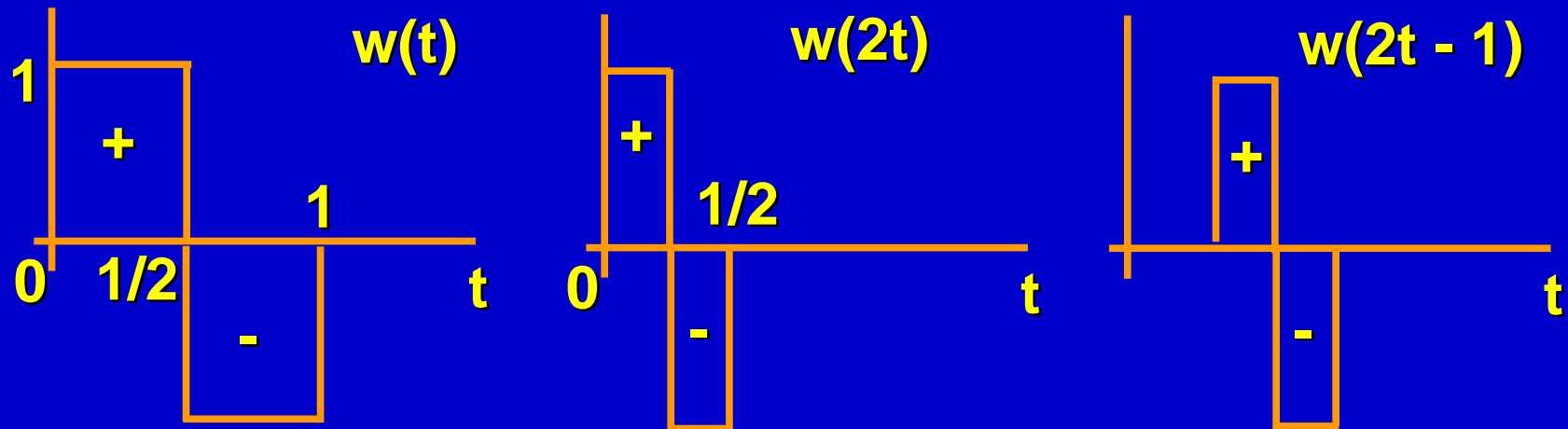
2. Scaling function is orthogonal to wavelet:



$$\int \phi(t) w(t) dt = 0$$

Reason: +ve and -ve areas cancel each other.

3. Wavelet is orthogonal across scales:



$$\int w(t) w(2t) dt = 0, \quad \int w(t) w(2t - 1) dt = 0$$

Reason: finer scale versions change sign while coarse scale version remains constant.

Wavelet Bases

Our goal is to use $w(t)$, its scaled versions (dilations) and their shifts (translates) as building blocks for continuous-time functions, $f(t)$. Specifically, we are interested in the class of functions for which we can define the inner product:

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t) g^*(t) dt < \infty$$

Such functions $f(t)$ must have finite energy:

$$\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$$

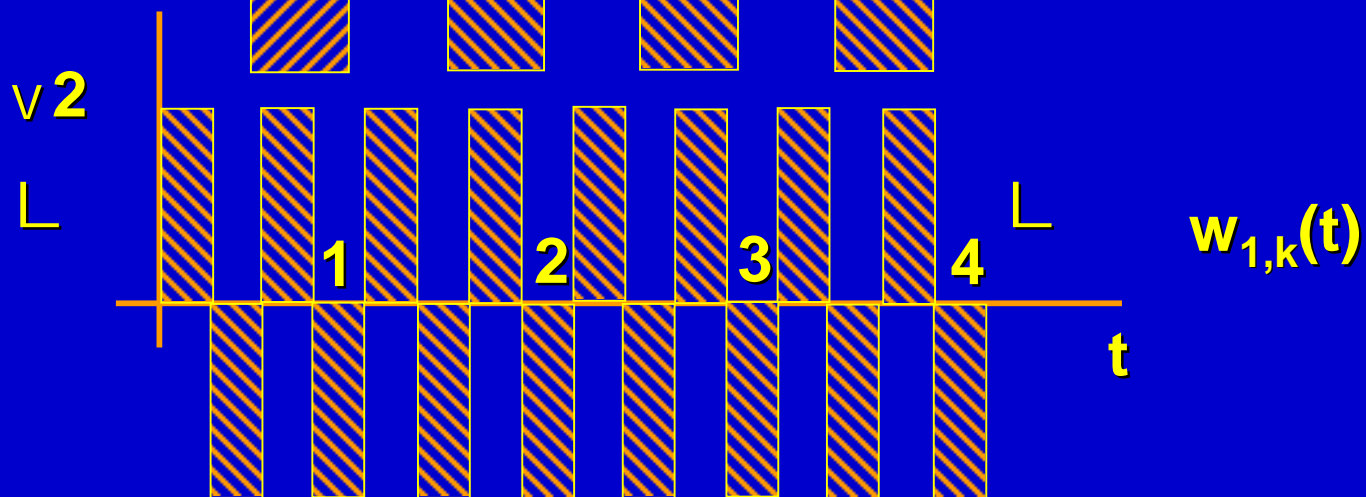
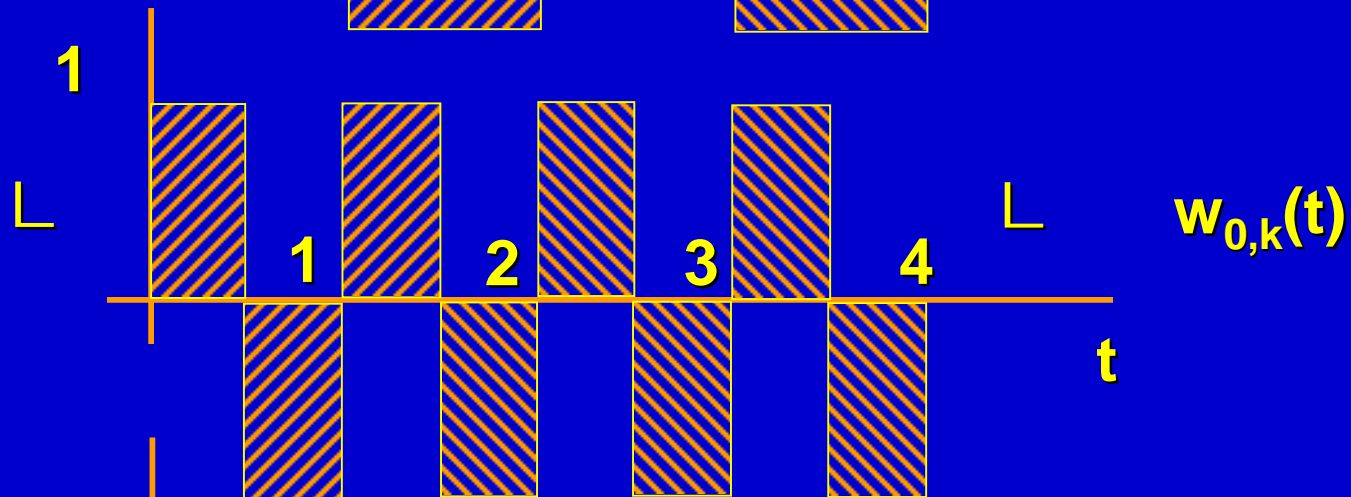
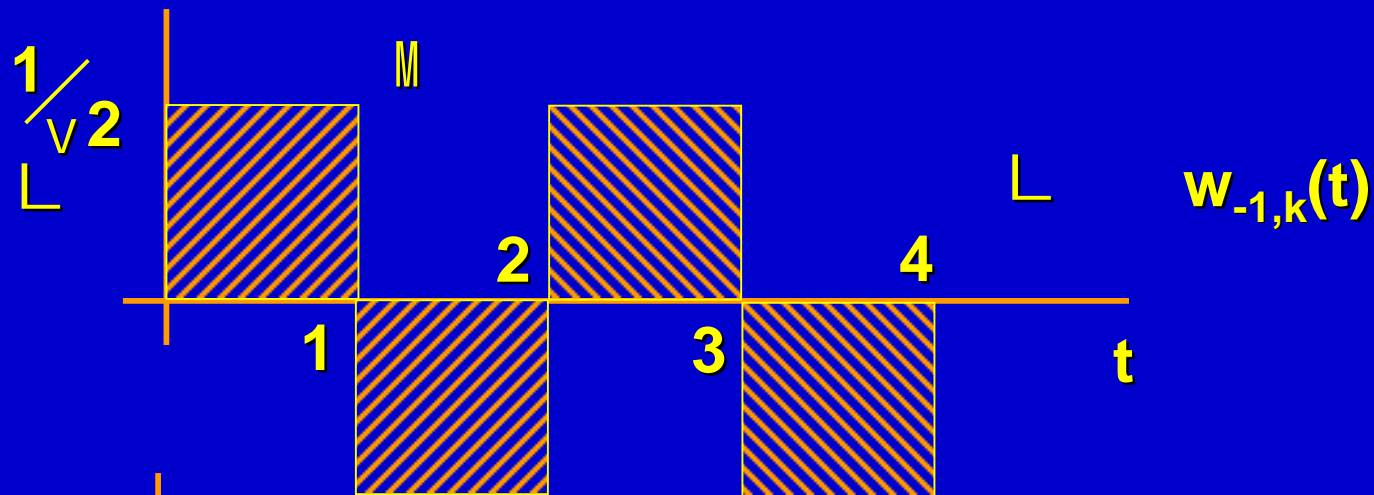
and they are said to belong to the Hilbert space, $L^2(\mathfrak{R})$.

Consider all dilations and translates of the Haar wavelet:

$$w_{j,k}(t) = 2^{j/2} w(2^j t - k) \quad ; \quad -\infty \leq j \leq \infty$$
$$-\infty \leq k \leq \infty$$

↑
Normalization factor so that $\|w_{j,k}(t)\| = 1$

$$\begin{aligned} \int w_{j,k}(t) w_{J,K}(t) dt &= \int 2^{j/2} w(2^j t - k) \cdot 2^{J/2} w(2^J t - K) dt \\ &= \begin{cases} 1 & \text{if } j = J \text{ and } k = K \\ 0 & \text{otherwise} \end{cases} \\ &= \delta[j - J] \delta[k - K] \end{aligned}$$



$w_{jk}(t)$ form an orthonormal basis for $L^2(\mathcal{R})$.

$$f(t) = \sum_{j,k} b_{jk} w_{jk}(t) ; \quad w_{jk}(t) = 2^{j/2} w(2^j t - k)$$

$$b_{jk} = \int_{-\infty}^{\infty} f(t) w_{jk}(t) dt$$

Multiresolution Analysis

Key ingredients:

1. A sequence of embedded subspaces:

$$\{0\} \subset \dots \subset V_{-1} \subset V_0 \subset V_1 \subset \dots \subset V_j \subset V_{j+1} \subset \dots \subset L^2(\mathfrak{R})$$

$L^2(\mathfrak{R})$ = all functions with finite energy

$$= \{f(t) : \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty\} \quad \text{Hilbert space}$$

Requirements:

- **Completeness as $j \rightarrow \infty$.** If $f(t)$ belongs to $L^2(\mathfrak{R})$ and $f_j(t)$ is the portion of $f(t)$ that lies in V_j , then $\lim_{j \rightarrow \infty} f_j(t) = f(t)$

Restated as a condition on the subspaces:

$$\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(\mathfrak{R})$$

- **Emptiness as $j \rightarrow -\infty$**

$$\lim_{j \rightarrow -\infty} \|f_j(t)\| = 0$$

Restated as a condition on the subspaces:

$$\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$$

**2. A sequence of complementary subspaces, W_j ,
such that $V_j + W_j = V_{j+1}$**

and $V_j \cap W_j = \{0\}$ (no overlap)

This is written as

$$V_j \oplus W_j = V_{j+1} \text{ (Direct sum)}$$

**Note: An orthogonal multiresolution will have W_j
orthogonal to V_j : $W_j \perp V_j$.**

So orthogonality will ensure that $V_j \cap W_j = \{0\}$

We thus have

$$V_1 = V_0 \oplus W_0$$

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$$

$$V_3 = V_2 \oplus W_2 = V_0 \oplus W_0 \oplus W_1 \oplus W_2$$

⋮

$$V_J = V_{J-1} \oplus W_{J-1} = V_0 \oplus \sum_{j=0}^{J-1} W_j$$

⋮

$$L^2(\mathfrak{R}) = V_0 \oplus \sum_{j=0}^{\infty} W_j$$

We can also write the recursion for $j < 0$

$$V_0 = V_{-1} \oplus W_{-1}$$

$$= V_{-2} \oplus W_{-2} \oplus W_{-1}$$

⋮

$$= V_{-k} \oplus \sum_{j=-k}^{-1} W_j$$

⋮

$$= \sum_{j=-\infty}^{-1} W_j$$

$$\Rightarrow L^2(\mathfrak{R}) = \sum_{j=-\infty}^{\infty} W_j$$

3. A scaling (dilation) law:

If $f(t) \in V_j$ then $f(2t) \in V_{j+1}$

4. A shift (translation) law:

If $f(t) \in V_j$ then $f(t-k) \in V_j$ **k integer**

5. V_0 has a shift-invariant basis, $\{\phi(t-k) : -\infty \leq k \leq \infty\}$

W_0 has a shift-invariant basis, $\{w(t-k) : -\infty \leq k \leq \infty\}$

We expect that $V_1 = V_0 + W_0$ will have twice as many basis functions as V_0 alone.

First possibility: $\{\phi(t-k), w(t-k) : -\infty \leq k \leq \infty\}$

Second possibility: use the scaling law i.e.

if $\phi(t-k) \in V_0$, then $\phi(2t-k) \in V_1$

So

V_1 has a shift-invariant basis, $\{\sqrt{2} \phi(2t-k): -\infty \leq k \leq \infty\}$

Can we relate this basis for V_1 to the basis for V_0 ?

We know that

$$V_0 \subset V_1$$

So any function in V_0 can be written as a combination of the basic functions for V_1 .

In particular, since $\phi(t) \in V_0$, we can write

$$\phi(t) = 2 \sum_k h_0[k] \phi(2t - k)$$

This is the Refinement Equation (a.k.a. the Two-Scale Difference Equation or the Dilation Equation).

We also know that

$$W_0 = V_1 - V_0$$

So

$$W_0 \subset V_1$$

This means that any function in W_0 can also be written as a combination of the basic functions for V_1 .

Since $w(t) \in W_0$, we can write

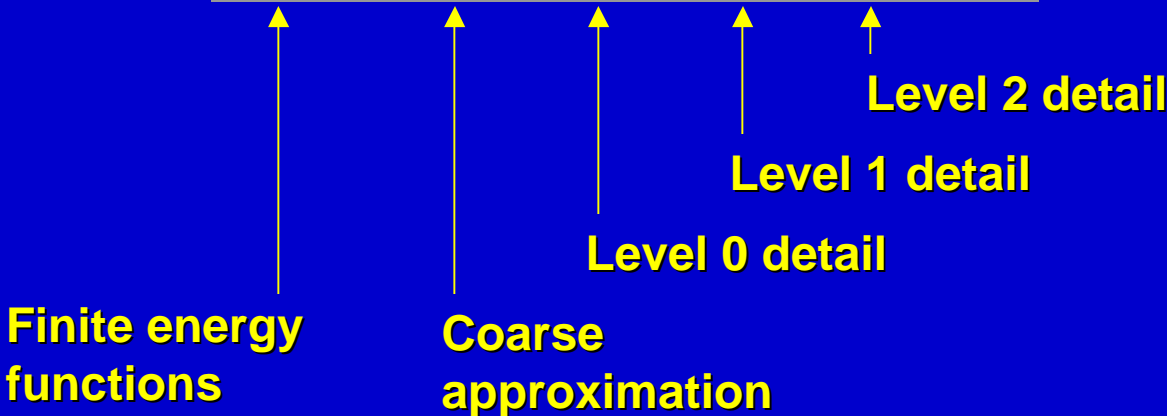
$$w(t) = 2 \sum_k h_1[k] \phi(2t - k)$$

**Wavelet
Equation**

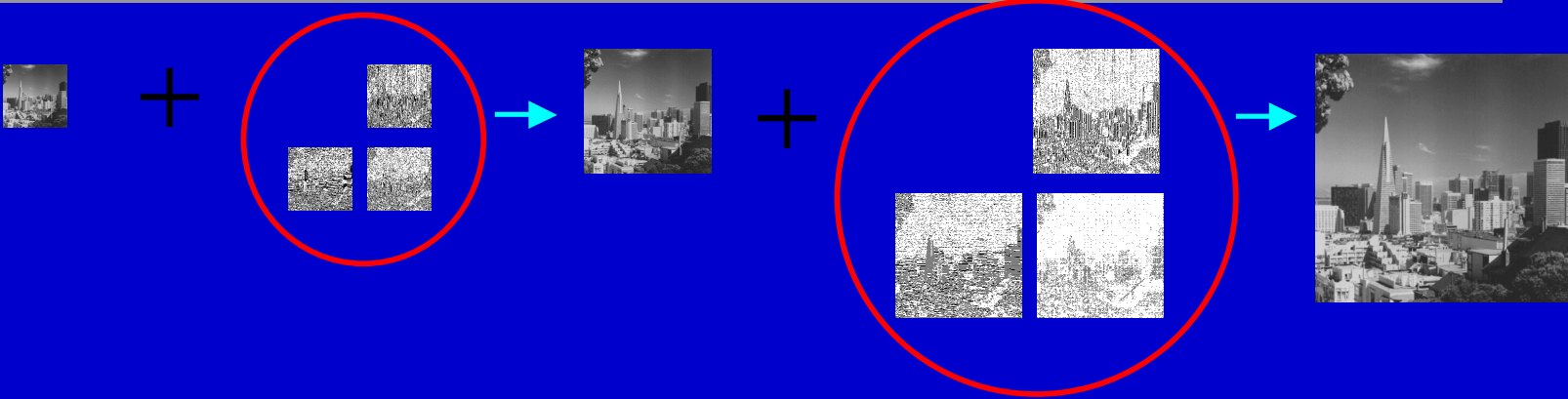
Multiresolution Representations

Functions:

$$L^2(\mathcal{R}) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \dots$$



Images:



Multiresolution Representations

Geometry:



N = 34, Level = 3



N = 130, Level = 4



N = 514, Level = 5



N = 2050, Level = 6



N = 8194, Level = 7



N = 32770, Level = 8

Course 18.327 and 1.130

Wavelets and Filter Banks

Refinement Equation: Iterative and Recursive Solution Techniques; Infinite Product Formula; Filter Bank Approach for Computing Scaling Functions and Wavelets

Solution of the Refinement Equation

$$\phi(t) = \sum_{k=0}^N h_0[k] \phi(2t-k)$$

First, note that the solution to this equation may not always exist! The existence of the solution will depend on the discrete-time filter $h_0[k]$.

If the solution does exist, it is unlikely that $\phi(t)$ will have a closed form solution. The solution is also unlikely to be smooth. We will see, however, that if $h_0[n]$ is FIR with

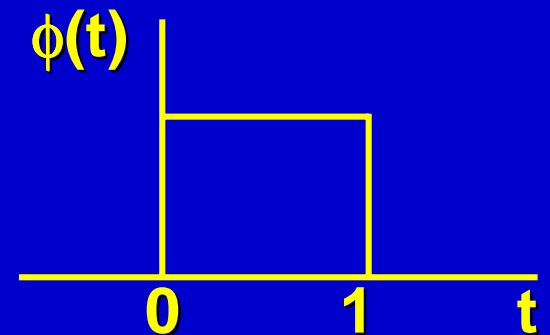
$$h_0[n] = 0 \text{ outside } 0 \leq n \leq N$$

then $\phi(t)$ has compact support:

$$\phi(t) = 0 \text{ outside } 0 < t < N$$

Approach 1 Iterate the box function

$$\phi^{(0)}(t) = \text{box function on } [0, 1]$$



$$\phi^{(i+1)}(t) = 2 \sum_{k=0}^N h_0[k] \phi^{(i)}(2t - k)$$

If the iteration converges, the solution will be given by

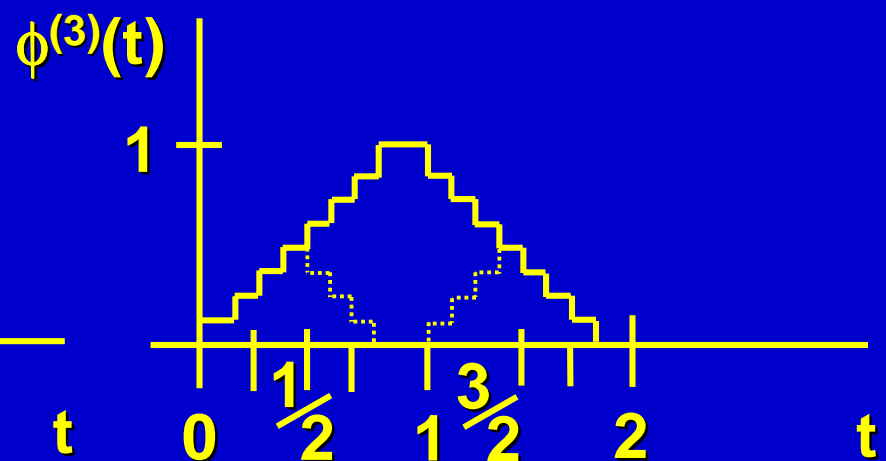
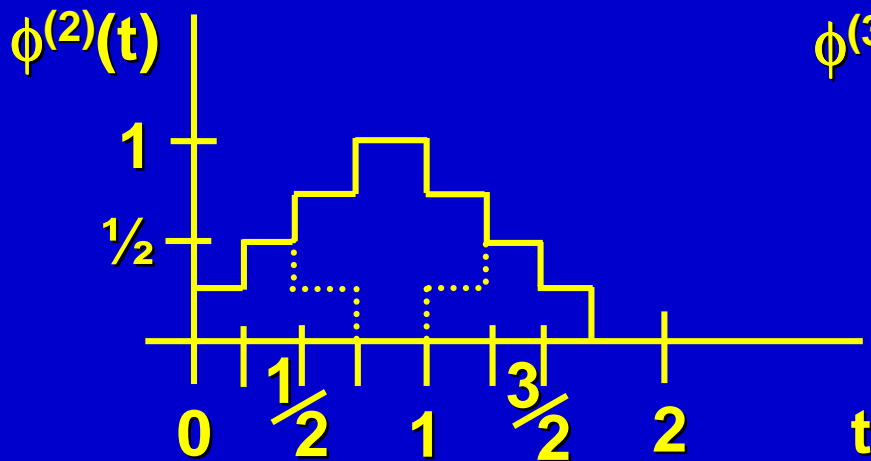
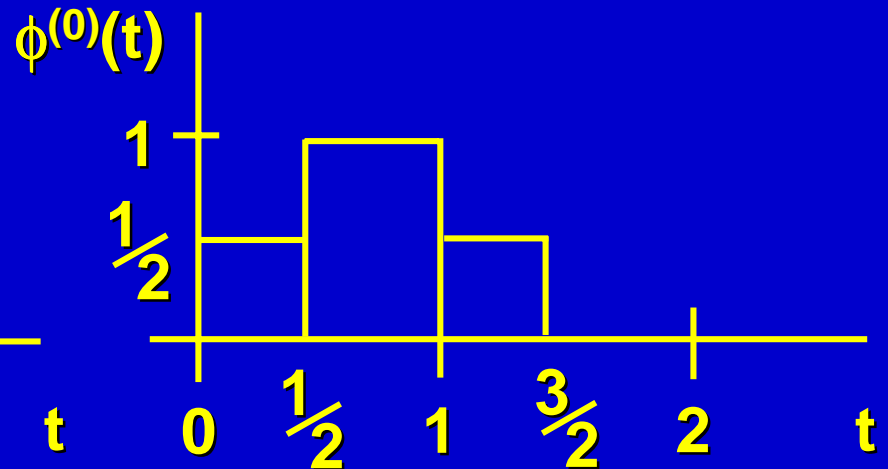
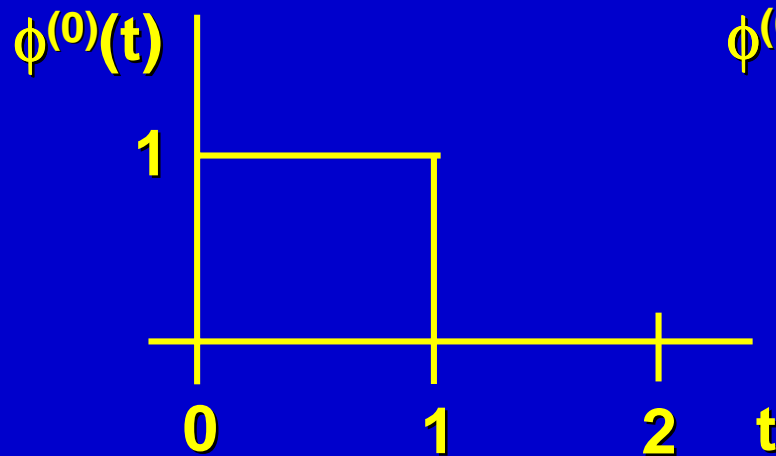
$$\lim_{i \rightarrow \infty} \phi^{(i)}(t)$$

This is known as the **cascade algorithm**.

Example: suppose $h_0[k] = \{1/4, 1/2, 1/4\}$

$$\phi^{(i+1)}(t) = \frac{1}{2} \phi^{(i)}(2t) + \phi^{(i)}(2t - 1) + \frac{1}{2} \phi^{(i)}(2t - 2)$$

Then



Converges to the hat function on $[0, 2]$

Approach 2 Use recursion

First solve for the values of $\phi(t)$ at integer values of t .

Then solve for $\phi(t)$ at half integer values, then at quarter integer values and so on.

This gives us a set of discrete values of the scaling function at all dyadic points $t = n/2^i$.

At integer points:

$$\phi(n) = 2 \sum_{k=0}^N h_0[k] \phi(2n - k)$$

Suppose $N = 3$

$$\phi(0) = 2 \sum_{k=0}^3 h_0[k] \phi(-k)$$

$$\phi(1) = 2 \sum_{k=0}^3 h_0[k] \phi(2-k)$$

$$\phi(2) = 2 \sum_{k=0}^3 h_0[k] \phi(4-k)$$

$$\phi(3) = 2 \sum_{k=0}^3 h_0[k] \phi(6-k)$$

Using the fact that $\phi(n) = 0$ for $n < 0$ and $n > N$, we can write this in matrix form as

$$\begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix} = 2 \begin{bmatrix} h_0[0] & & & \\ h_0[2] & h_0[1] & h_0[0] & \\ & h_0[3] & h_0[2] & h_0[1] \\ & & & h_0[3] \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix}$$

Notice that this is an eigenvalue problem

$$\lambda\Phi = A\Phi$$

where the eigenvector is the vector of scaling function values at integer points and the eigenvalue is $\lambda = 1$.

Note about normalization:

**Since $(A - \lambda I)\Phi = 0$ has a non-unique solution, we must choose an appropriate normalization for Φ
The correct normalization is**

$$\sum_n \phi(n) = 1$$

This comes from the fact that we need to satisfy the partition of unity condition, $\sum_n \phi(x-n) = 1$.

At half integer points:

$$\phi(n/2) = 2 \sum_{k=0}^N h_0[k] \phi(n-k)$$

So, for $N = 3$, we have

$$\begin{bmatrix} \phi(1/2) \\ \phi(3/2) \\ \phi(5/2) \end{bmatrix} = 2 \begin{bmatrix} h_0[1] & h_0[0] & & & \\ h_0[3] & h_0[2] & h_0[1] & h_0[0] & \\ & & & & \\ & & & & \\ & & & h_0[3] & h_0[2] \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix}$$

Scaling Relation and Wavelet Equation in Frequency Domain

$$\phi(t) = 2 \sum_k h_0[k] \phi(2t - k)$$

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(t) e^{-i\Omega t} dt &= 2 \sum_k h_0[k] \int_{-\infty}^{\infty} \phi(2t - k) e^{-i\Omega t} dt \\ &= 2 \sum_k h_0[k] \frac{1}{2} \int_{-\infty}^{\infty} \phi(\tau) e^{-i\Omega(\tau + k)/2} d\tau \\ &= \sum_k h_0[k] e^{-i\Omega k/2} \int_{-\infty}^{\infty} \phi(\tau) e^{-i\Omega\tau/2} d\tau \end{aligned}$$

$$\begin{aligned}
 \text{i.e. } \hat{\phi}(\Omega) &= H_0\left(\frac{\Omega}{2}\right) \cdot \hat{\phi}\left(\frac{\Omega}{2}\right) \\
 &= H_0\left(\frac{\Omega}{2}\right) \cdot H_0\left(\frac{\Omega}{4}\right) \cdot \hat{\phi}\left(\frac{\Omega}{4}\right) \\
 &\quad \vdots \\
 &= \left\{ \prod_{j=1}^{\infty} H_0\left(\frac{\Omega}{2^j}\right) \right\} \hat{\phi}(0)
 \end{aligned}$$

$$\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(t) dt = 1 \text{ (Area is normalized to 1)}$$

So

$$\hat{\phi}(\Omega) = \prod_{j=1}^{\infty} H_0\left(\frac{\Omega}{2^j}\right) \quad \text{Infinite Product Formula}$$

Similarly

$$w(t) = 2 \sum_k h_1[k] \phi(2t - k)$$

leads to

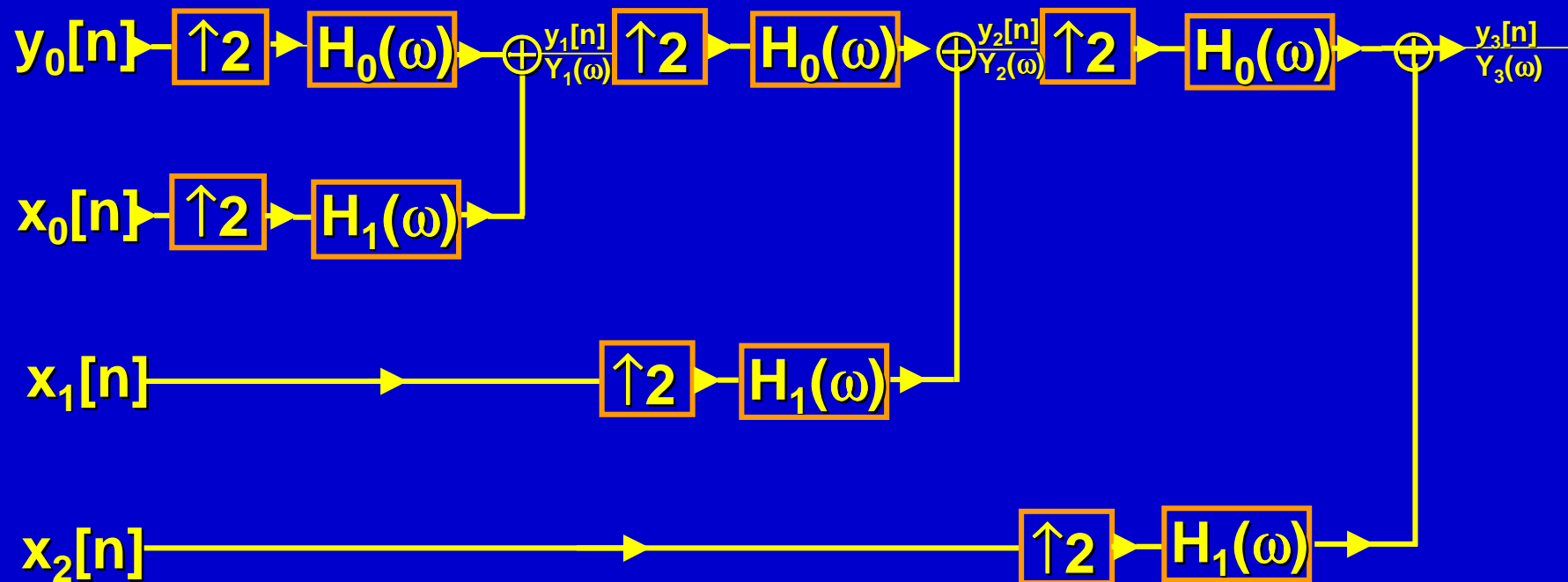
$$\hat{w}(\Omega) = H_1\left(\frac{\Omega}{2}\right) \hat{\phi}\left(\frac{\Omega}{2}\right)$$

Desirable properties for $H_0(\omega)$:

- $H(0) = 1$, so that $\hat{\phi}(0) = 1$
- $H(\omega)$ should decay to zero as $\omega \rightarrow \pi$,

$$\text{so that } \int_{-\infty}^{\infty} |\hat{\phi}(\Omega)|^2 d\Omega < \infty$$

Computation of the Scaling Function and Wavelet – Filter Bank Approach



Normalize so that $\sum_n h_0[n] = 1$.

i. Suppose $y_0[n] = \delta[n]$ and $x_k[n] = 0$.

$$Y_0(\omega) = 1$$

$$Y_1(\omega) = Y_0(2\omega) H_0(\omega) = H_0(\omega)$$

$$Y_2(\omega) = Y_1(2\omega) H_0(\omega) = H_0(2\omega) H_0(\omega)$$

$$Y_3(\omega) = Y_2(2\omega) H_0(\omega) = H_0(4\omega) H_0(2\omega) H_0(\omega)$$

After K iterations:

$$Y_K(\omega) = \prod_{k=0}^{K-1} H_0(2^k \omega)$$

What happens to the sampling period?

Sampling period at input = $T_0 = 1$ (say)

Sampling period at output = $T_K = 1/2^K$

Treat the output as samples of a continuous time signal, $y_K^c(t)$, with sampling period $1/2^K$:

$$y_K[n] = \frac{1}{2^K} y_K^c(n/2^K)$$

$$\Rightarrow Y_K(\omega) = \hat{Y}_K^c(2^K\omega) \quad ; \quad -\pi \leq \omega \leq \pi$$

($y_K^c(t)$ is chosen to be bandlimited)

Replace $2^K\omega$ with Ω :

$$\hat{Y}_K^c(\Omega) = Y_K(\Omega/2^K) = \prod_{k=0}^{K-1} H_0(\Omega/2^{K-k}) = \prod_{j=1}^K H_0(\Omega/2^j) \quad ;$$
$$-2^K\pi \leq \Omega \leq 2^K\pi$$

So

$$\lim_{K \rightarrow \infty} \hat{Y}_K^c(\Omega) = \prod_{j=1}^{\infty} H_0(\Omega/2^j) = \hat{\phi}(\Omega)$$

⇒ $2^K y_K[n]$ converges to the samples of the scaling function, $\phi(t)$, taken at $t = n/2^K$.

ii. Suppose $y_0[n] = 0$, $x_0[n] = \delta[n]$ and all other $x_k[n] = 0$

$$Y_K(\omega) = H_1(2^{K-1}\omega) \prod_{k=0}^{K-2} H_0(2^k\omega)$$

Then

$$\begin{aligned} \hat{Y}_K^c(\Omega) &= Y_K(\Omega/2^K) = H_1\left(\frac{\Omega}{2}\right) \prod_{k=0}^{K-2} H_0(\Omega/2^{K-k}) \\ &= H_1\left(\frac{\Omega}{2}\right) \prod_{j=1}^{K-1} H_0\left(\frac{1}{2} \cdot \frac{\Omega}{2^j}\right) \end{aligned}$$

So

$$\lim_{K \rightarrow \infty} \hat{Y}_K^c(\Omega) = H_1(\Omega/2) \hat{\phi}(\Omega/2) = \hat{w}(\Omega)$$

⇒ $2^K y_K[n]$ converges to the samples of the wavelet, $w(t)$, taken at $t = n/2^K$.

Support of the Scaling Function



$$\text{length} \{v[n]\} = 2 \cdot \text{length} \{y_{k-1}[n]\} - 1$$

Suppose that

$$h_0[n] = 0 \text{ for } n < 0 \text{ and } n > N$$

$$\begin{aligned} \Rightarrow \text{length} \{y_k[n]\} &= \text{length} \{v[n]\} + \text{length} \{h_0[n]\} - 1 \\ &= 2 \cdot \text{length} \{y_{k-1}[n]\} + N - 1 \end{aligned}$$

Solve the recursion with $\text{length} \{y_0[n]\} = 1$

So

$$\text{length} \{y_k[n]\} = (2^k - 1)N + 1$$

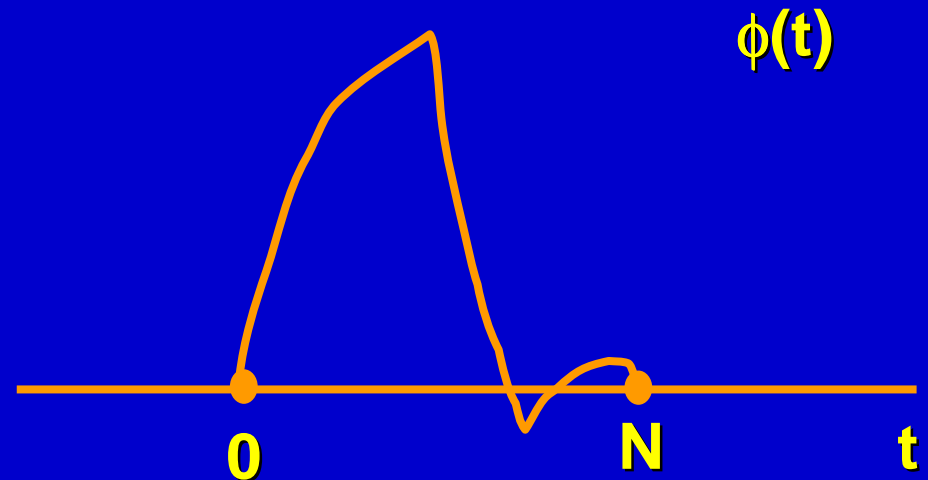
$$\text{i.e. length } \{y_K^c(t)\} = T_K \cdot \text{length } \{y_K[n]\}$$

$$= \frac{(2^K - 1)N + 1}{2^K}$$

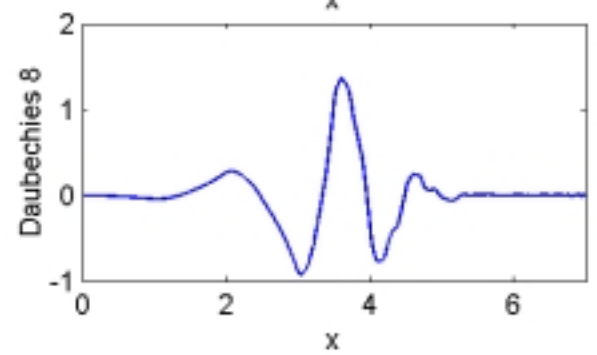
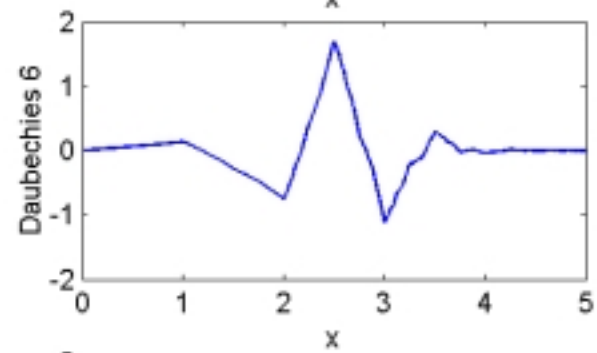
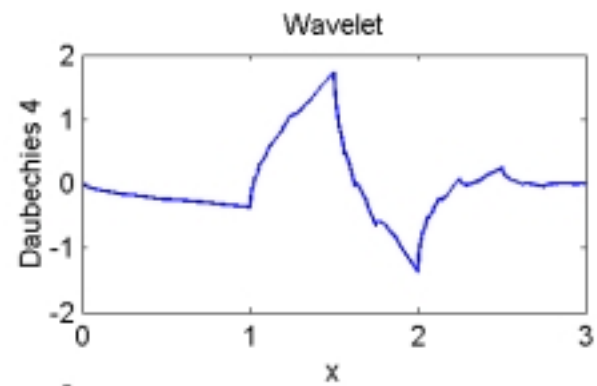
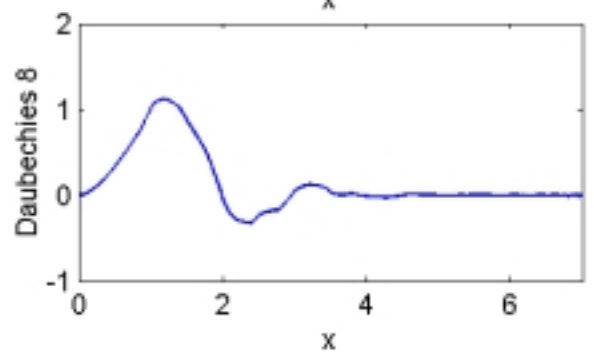
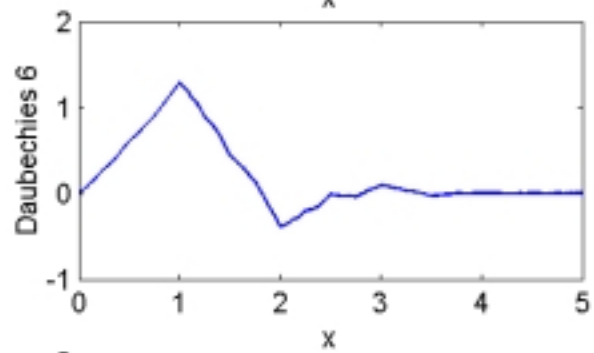
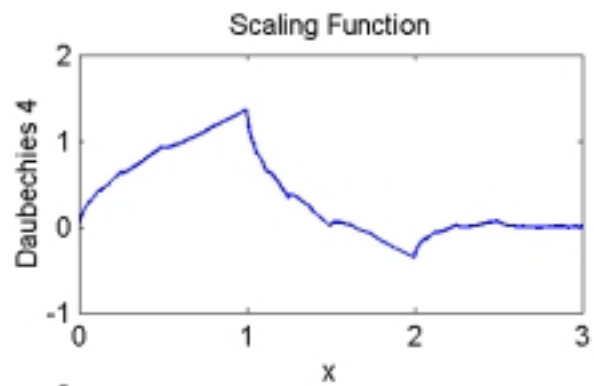
$$= N - \frac{N-1}{2^K}$$

$$\lim_{K \rightarrow \infty}$$

$$\text{length } \{\phi(t)\} = N$$



So the scaling function is supported on the interval $[0, N]$



Matlab Example 6

Generation of orthogonal scaling
functions and wavelets

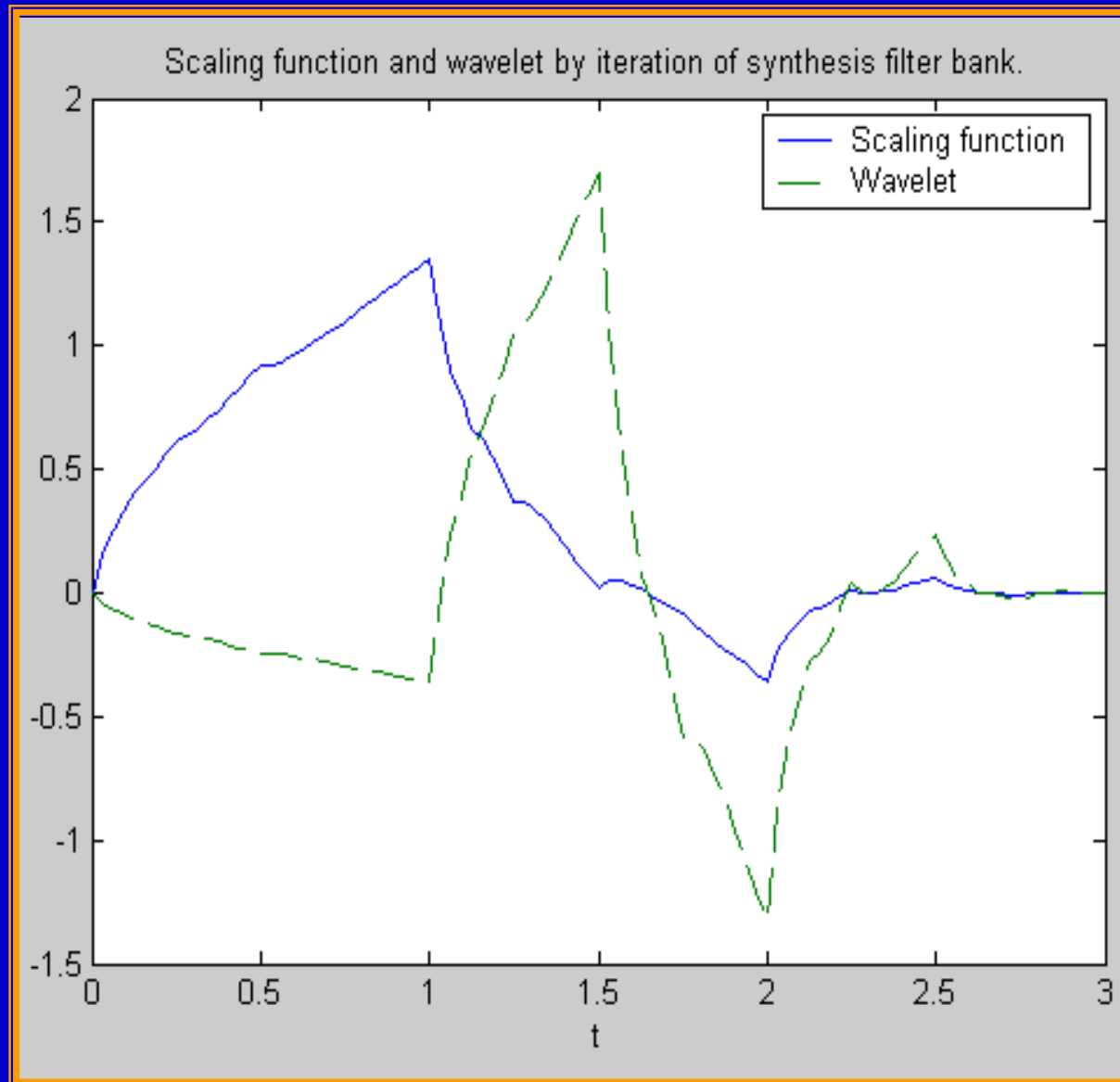


MATLAB M-file

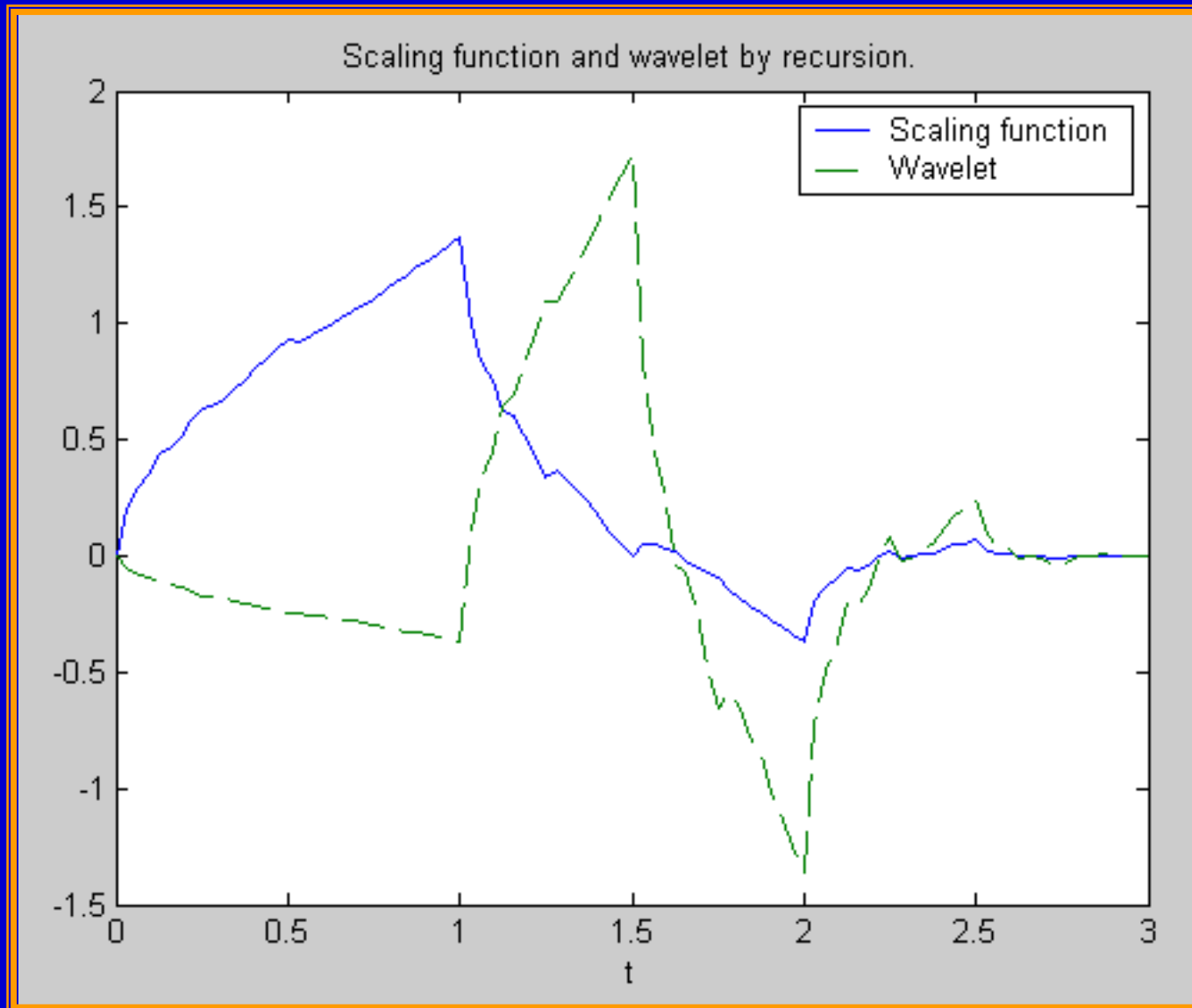


MATLAB M-file

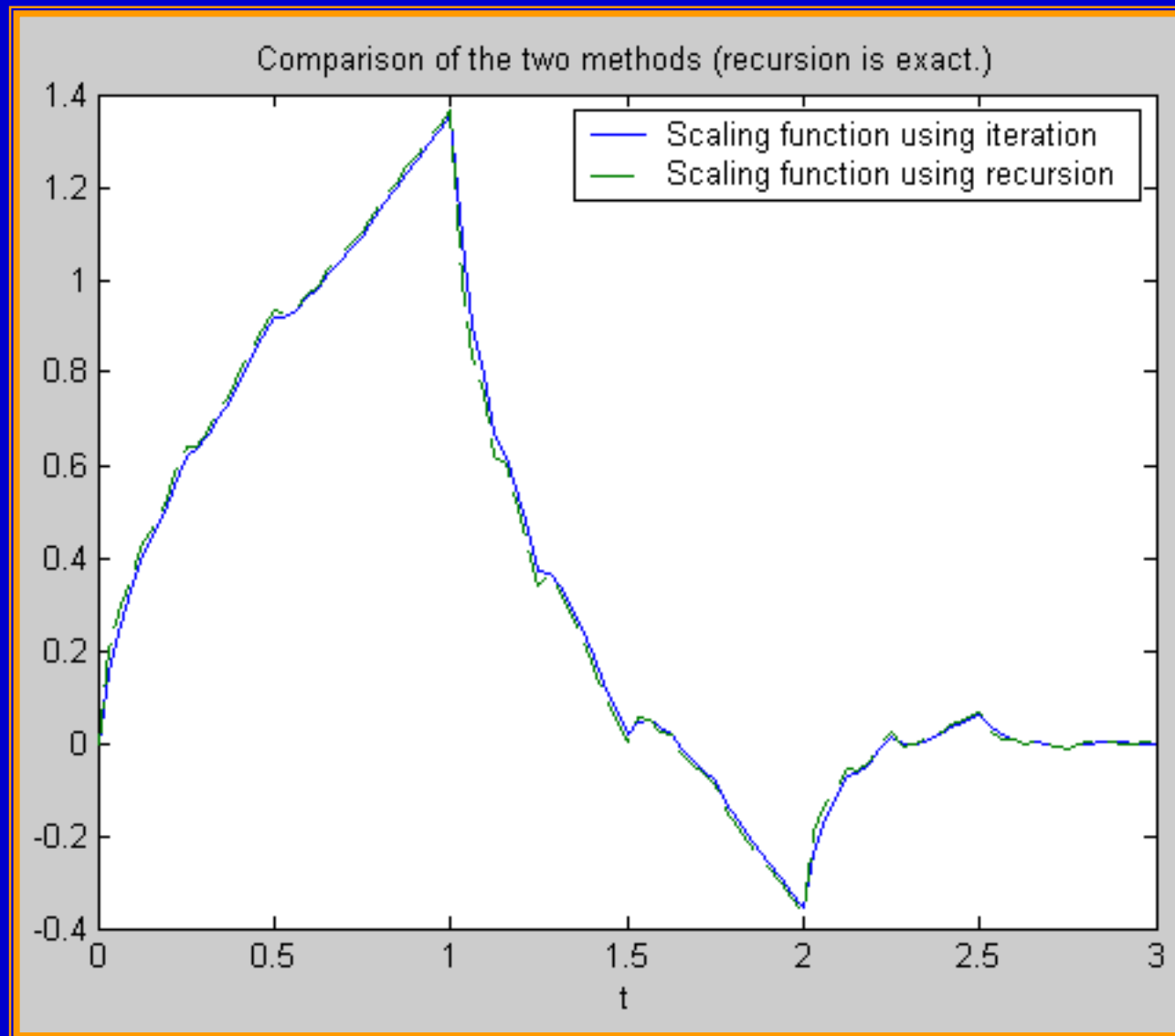
By Inverse DWT



By Recursion



Comparison



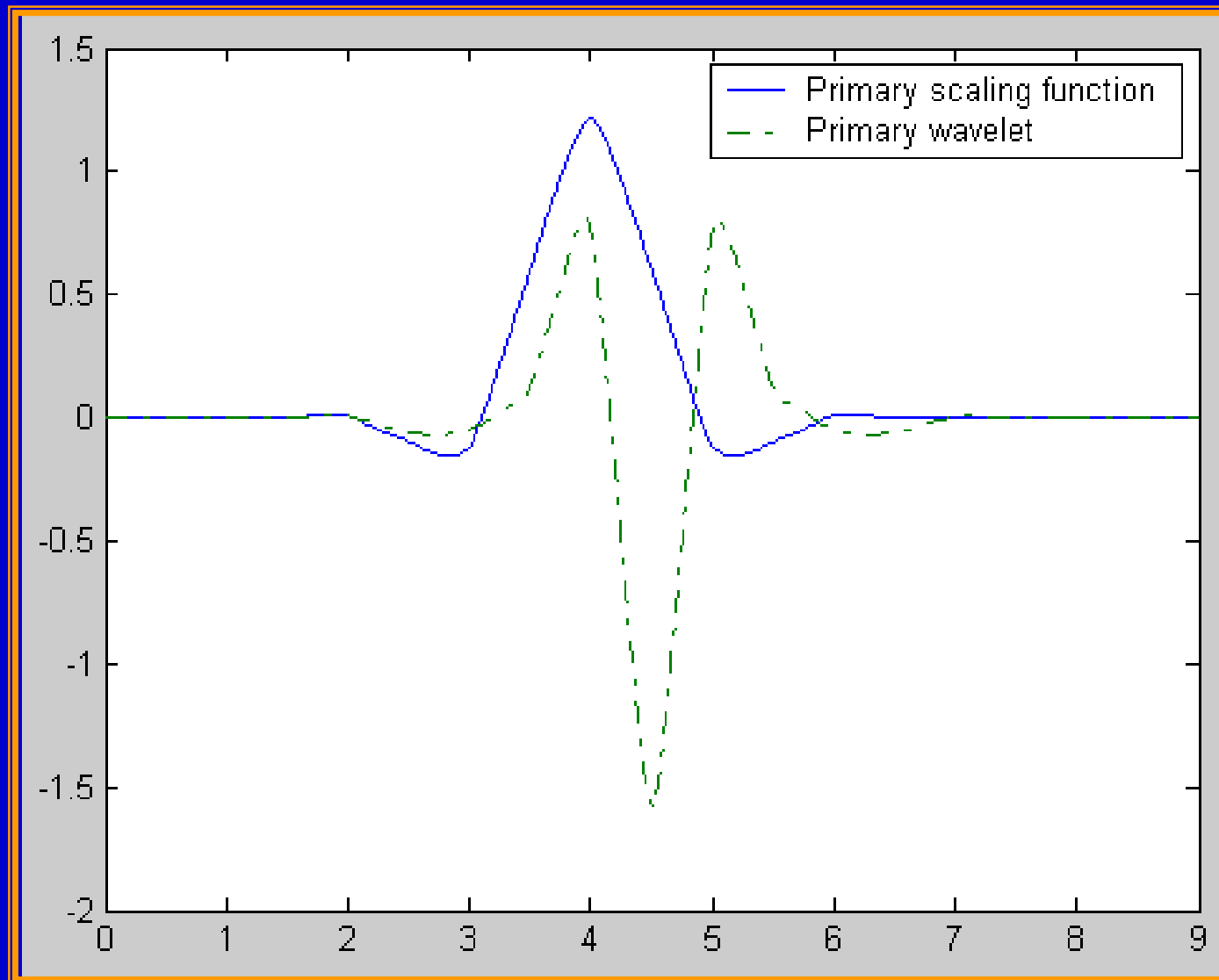
Matlab Example 7

Generation of biorthogonal scaling functions and wavelets.



MATLAB M-file

Primary Daub 9/7 Pair



Dual Daub 9/7 Pair

