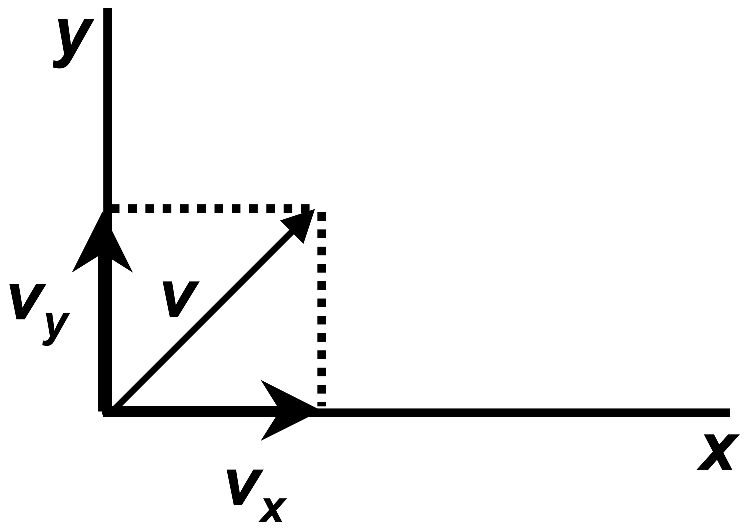
Course 18.327 and 1.130
Wavelets and Filter Banks

Orthogonal wavelet bases: connection to orthogonal filters; orthogonality in the frequency domain. Biorthogonal wavelet bases.

Orthogonal Wavelets

2D Vector Space:

Basis vectors are *i*, *j* $\rightarrow$ orthonormal basis



v_x and v_y are the projections of *v* onto the *x* and *y* axes:

$$v_x = \langle v , i \rangle \; i$$

$$v_y = \langle v , j \rangle \; j$$

$$\langle \boldsymbol{v}, \boldsymbol{i} \rangle = [v_x \quad v_y] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = v_x$$ **Inner Product**

Orthogonal multiresolution spaces:

V_j **has an orthonormal basis** $\{\underbrace{2^{j/2}\,\phi(2^j t - k)}_{\phi_{j,k}(t)} : -\infty \le k \le \infty\}$

W_j **has an orthonormal basis** $\{\underbrace{2^{j/2}\,w(2^j t - k)}_{w_{j,k}(t)} : -\infty \le k \le \infty\}$

Orthonormal means

$$\langle \phi_{j,k}(t)\ ,\ \phi_{j,l}(t) \rangle = \int_{-\infty}^{\infty} 2^{j/2}\phi(2^j t - k)\ 2^{j/2}\phi(2^j t - l)dt = \delta[k - l]$$

$$\langle w_{j,k}(t)\ ,\ w_{j,l}(t) \rangle = \int_{-\infty}^{\infty} 2^{j/2}w(2^j t - k)\ 2^{j/2}w(2^j t - l)dt = \delta[k - l]$$

For orthogonal multiresolution spaces, we have

$$V_j \perp W_j$$

So

$$\langle \phi_{j,k}(t)\ ,\ w_{j,l}(t) \rangle = 0$$

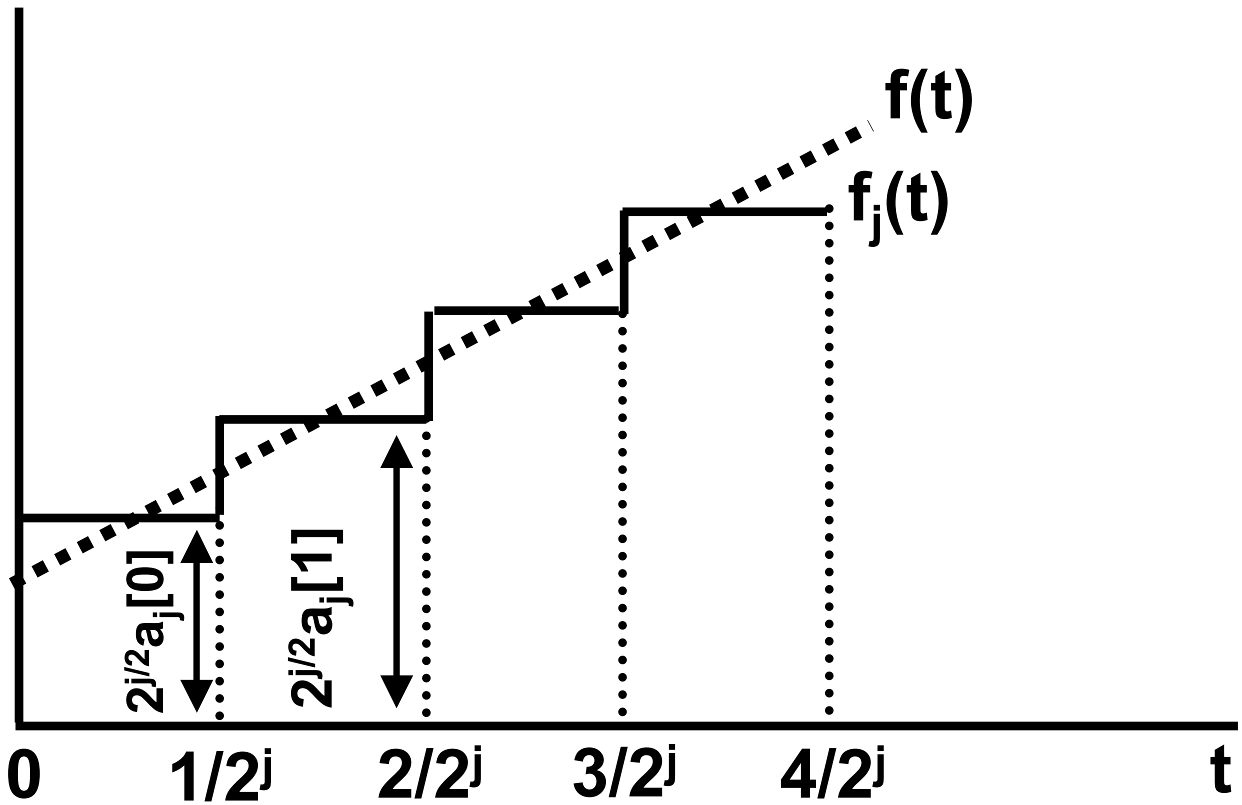
Projection of an L^2 function, f(t), onto V_j:

$$f_j(t) = \sum_k a_j[k]\, \phi_{j,k}(t)$$

with

$$a_j[k] = \langle f(t), \phi_{j,k}(t) \rangle$$

Haar example



Projection of f(t) onto W_j:

$$g_j(t) = \sum_k b_j[k] w_{j,k}(t)$$

with

$$b_j[k] = \langle f(t), w_{j,k}(t) \rangle$$

Biorthogonal Wavelet Bases

Two scaling functions and two wavelets:

Synthesis:

$$\phi(t) = 2 \sum_k f_0[k]\, \phi(2t - k)$$

$$w(t) = 2 \sum_k f_1[k]\, \phi(2t - k)$$

Analysis:

$$\tilde{\phi}(t) = 2 \sum_k h_0[-k]\, \tilde{\phi}(2t - k)$$

$$\tilde{w}(t) = 2 \sum_k h_1[-k]\, \tilde{\phi}(2t - k)$$

Two sets of multiresolution spaces:

$\{0\} \subset \ldots V_0 \subset V_1 \subset \ldots \subset V_j \subset \ldots \subset L^2(\square)$

$\{0\} \subset \ldots \tilde{V}_0 \subset \tilde{V}_1 \subset \ldots \subset \tilde{V}_j \subset \ldots \subset L^2(\square)$

$$V_j + W_j = V_{j+1}$$

$$\tilde{V}_j + \tilde{W}_j = \tilde{V}_{j+1}$$

Spaces are orthogonal w.r.t. each other i.e.

$$V_j \;\; \underline{\;h\;} \;\; \tilde{W}_j \qquad\qquad \tilde{V}_j \;\; \underline{\;h\;} \;\; W_j$$

V_0 **has a basis** $\{\phi(t-k) : -\infty < k < \infty\}$

$\tilde{V}_0$ **has a basis** $\{\tilde{\phi}(t-k) : -\infty < k < \infty\}$

W_0 **has a basis** $\{w(t-k): -\infty < k < \infty\}$

$\tilde{W}_0$ **has a basis** $\{\tilde{w}(t-k): -\infty < k < \infty\}$

Bases are orthogonal w.r.t. each other i.e.

$$\int \phi(t)\, \tilde{\phi}(t-k)\, dt = \delta[k] \qquad \int \phi(t)\, \tilde{w}(t-k)\, dt = 0$$

$$\int w(t)\, \tilde{\phi}(t-k)\, dt = 0 \qquad \int w(t)\, \tilde{w}(t-k)\, dt = \delta[k]$$

Equivalent to perfect reconstruction conditions on filters

Representation of functions in a biorthogonal basis:

$$f(t) = \sum_{k} c_k \phi(t-k) + \sum_{j=0}^{\infty} \sum_{k} d_{j,k}\, 2^{j/2}\, w(2^j t - k)$$

$$c_k = \int f(t)\, \tilde{\phi}(t-k)\, dt$$

$$d_{j,k} = 2^{j/2} \int f(t)\, \tilde{w}(2^j t - k)\, dt$$

Similarly, we can represent f(t) in the dual basis

$$f(t) = \sum_k \tilde{c}_k \tilde{\phi}(t - k) + \sum_{j=0}^{\infty} \sum_k \tilde{d}_{j,k} 2^{j/2} \tilde{w}(2^j t - k)$$

$$\tilde{c}_k = \int f(t)\, \phi(t - k)\, dt$$

$$\tilde{d}_{j,k} = 2^{j/2} \int f(t)\, w(2^j t - k)\, dt$$

Note: When $f_0[k] = h_0[-k]$ and $f_1[k] = h_1[-k]$, we have

$$\phi(t) = \tilde{\phi}(t) \Rightarrow V_j = \tilde{V}_j$$

$$w(t) = \tilde{w}(t) \Rightarrow W_j = \tilde{W}_j$$

i.e. we have orthogonal wavelets!

Connection between orthogonal wavelets and orthogonal filters

Start with the orthonormality requirement on scaling functions

$$\delta[n] = \int_{-\infty}^{\infty} \phi(t)\, \phi(t - n)\, dt$$

And then change scale using the refinement equation:

$$\delta[n] = \int_{-\infty}^{\infty} 2 \sum_k h_0[k]\phi(2t - k)\; 2 \sum_l h_0[l]\phi(2(t - n)-l)\, dt$$

$$= 4 \sum_k h_0[k] \sum_l h_0[l] \int_{-\infty}^{\infty} \phi(\tau)\, \phi(\tau + k - 2n - l)\, d\tau/2$$

$$= 2 \sum_k h_0[k] \sum_l h_0[l]\, \delta[-k + 2n + l]$$

$$\boxed{\delta[n] = 2 \sum_k h_0[k]\, h_0[k - 2n]}$$

This is the "double shift" orthogonality condition that we previously encountered for orthogonal filters.

So orthogonal filters are necessary for orthogonal wavelets. Are they sufficient?

Consider the cascade algorithm

$$\phi^{(i+1)}(t) = 2\sum_{k=0}^{N} h_0[k]\phi^{(i)}(2t-k)$$

If the filters are orthogonal and

$$\int_{-\infty}^{\infty} \phi^{(i)}(t)\,\phi^{(i)}(t-n)dt = \delta[n]$$

then

$$\int_{-\infty}^{\infty} \phi^{(i+1)}(t)\,\phi^{(i+1)}(t-n)dt = \delta[n]$$

Our initial guess for the iteration was $\phi^{(0)}(t)$ = box on [0,1] which is orthonormal w.r.t. its shifts. By induction, we have $\int_{-\infty}^{\infty} \phi(t)\,\phi(t-n)dt = \delta[n]$ as the limit, provided that the cascade algorithm converges.

Note: with the alternating flip requirement, which was necessary for the highpass filter in the case of orthogonal filters, we can show that

$$\int_{-\infty}^{\infty} w(t)\, w(t-n)\,dt = \delta[n]$$

and

$$\int_{-\infty}^{\infty} \phi(t)\, w(t-n)\,dt = 0$$

Orthogonality in the Frequency Domain

Let

$$a[n] = \int_{-\infty}^{\infty} \phi(t)\, \phi(t-n)\,dt$$

Use Parseval's theorem

$$a[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}(\Omega)\, \overline{\hat{\phi}(\Omega)\, e^{-i\Omega n}}\, d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{\phi}(\Omega)\right|^2 e^{i\Omega n}\, d\Omega$$

Trick: split integral over entire Ω axis into a sum of integrals over 2π intervals

$$a[n] = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} \left|\hat{\phi}(\Omega + 2\pi k)\right|^2 e^{i(\Omega + 2\pi k)n}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} \left|\hat{\phi}(\Omega + 2\pi k)\right|^2 e^{i\Omega n}\, d\pi$$

Take the Discrete Time Fourier Transform of both sides

$$A(\omega) \equiv \sum_{n} a[n]e^{-i\omega n} = \sum_{k=-\infty}^{\infty} \left| \hat{\phi}(\omega + 2\pi k) \right|^2$$

If $\phi(t - n)$ are orthonormal, then $a[n] = \delta[n]$

$$\Rightarrow A(\omega) = 1$$

So the scaling function and it shifts are orthogonal if

$$\boxed{\sum_{k=-\infty}^{\infty} \left| \hat{\phi}(\omega + 2\pi k) \right|^2 = 1}$$

Note: if we set $\omega = 0$, then

$$\sum_{k} \left| \hat{\phi}(2\pi k) \right|^2 = 1$$

and since $\hat{\phi}(0) = 1$, we see that

$$\hat{\phi}(2\pi k) = 0 \text{ for } k \neq 0$$

Connection between orthogonal wavelets and orthogonal filters (in frequency domain):

Start with an orthogonal scaling function:

$$1 = \sum_{k=-\infty}^{\infty} \left| \hat{\phi}(\omega + 2\pi k) \right|^2$$

and then change scale using the refinement equation in the frequency domain:

$$\hat{\phi}(\Omega) = H_0(\Omega/2)\,\hat{\phi}(\Omega/2)$$

$$1 = \sum_{k=-\infty}^{\infty} \left| H_0(\Omega/2 + \pi k) \right|^2 \left| \hat{\phi}(\Omega/2 + \pi k) \right|^2$$

Trick: split sum into even and odd

$$1 = \sum_{k=-\infty}^{\infty} \left| H_0(\Omega/2 + 2\pi k) \right|^2 \left| \hat{\phi}(\Omega/2 + 2\pi k) \right|^2 +$$

$$\sum_{k=-\infty}^{\infty} \left| H_0(\Omega/2 + \pi(2k + 1)) \right|^2 \left| \hat{\phi}(\Omega/2 + \pi(2k + 1) \right|^2$$

$$= \left| H_0(\Omega/2) \right|^2 \sum_{k=-\infty}^{\infty} \left| \hat{\phi}(\Omega/2 + 2\pi k) \right|^2 +$$

$$\left| H_0(\Omega/2 + \pi) \right|^2 \sum_{k=-\infty}^{\infty} \left| \hat{\phi}(\Omega/2 + \pi + 2\pi k) \right|^2$$

But each of the two infinite sums is equal to 1

So the discrete time filter $H_0(\omega)$ must be orthogonal:

$$1 = \left| H_0(\omega) \right|^2 + \left| H_0(\omega + \pi) \right|^2$$