Day 1

1. Problem Set 1.3:

   Problems 1: Can a symmetric filter, with $h(k) = h(N-k)$, be a highpass filter?

   Problems 3: Which of the following filters are invertible? Find the inverse filters:

   a) $h(0) = \frac{2}{3}$ and $h(1) = -\frac{1}{3}$

   b) $h(0) = 2$ and $h(2) = 1$

   c) $h(n) = \frac{1}{n!}$ ($n = 0,1,2,\ldots$)

2. Problem Set 1.4:

   Problems 9: In a transmultiplexer, the synthesis bank comes before the analysis bank. Compute LL^T and LB^T and BB^T to verify that the Haar transmultiplexer still give perfect construction:

$$\begin{bmatrix} L \\ B \end{bmatrix} \begin{bmatrix} L^T & B^T \end{bmatrix} = \begin{bmatrix} LL^T & LB^T \\ BL^T & BB^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

   Problems 11: If $H_0(z) = 1$ and $H_1(z) = z^{-1}$ (no filtering) write the entries of $\begin{bmatrix} L \\ B \end{bmatrix} \begin{bmatrix} L^T & B^T \end{bmatrix} = I$.

3. Problem Set 3.3:

   Problems 2: Verify the theorem 3.6 directly:

   Theorem 3.6: the operator $(\downarrow L)$ and $(\uparrow M)$ commute if and only if L and M are relative prime. Their greatest common divisor is $(L, M) = 1$. In that case $(\uparrow L)(\downarrow M)x = (\downarrow M)(\uparrow L)x$ has components

$$u(n) = \begin{cases} x(Mk) & \text{if } n/L = k \text{ is an integer} \\ 0 & \text{if } n/L \text{ is not an integer} \end{cases}$$

   What happens to the odd-numbered components of x when we compute $(\uparrow 2)(\downarrow 2)x$?

2. Verifying (3.27): $\mathbf{v} = (\downarrow M)\mathbf{x}$ and $\mathbf{u} = (\uparrow L)\mathbf{v}$ have components:

$$\mathbf{v}(k) = \mathbf{x}(Mk)$$

$$\mathbf{u}(n) = \begin{cases} \mathbf{v}(k) = \mathbf{x}(Mk) = \mathbf{x}(\frac{Mn}{L}), & \text{if } \frac{n}{L} = k \text{ is integer} \\ 0, & \text{otherwise} \end{cases}$$

Change the order of $(\downarrow M)$ and $(\uparrow L)$: upsampling first puts in $L-1$ zeros between each $\mathbf{x}(n)$ and $\mathbf{x}(n+1)$. It has components:

$$\mathbf{v}'(n) = \begin{cases} \mathbf{x}(k) = \mathbf{x}(\frac{n}{L}), & \text{if } \frac{n}{L} = k \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$$

Downsampling $\mathbf{u}' = (\downarrow M)\mathbf{v}'$ keeps every Mth components of $\mathbf{v}'$ and removes all the other components:

$$\mathbf{u}'(n) = \mathbf{v}'(Mn) = \begin{cases} \mathbf{x}(k) = \mathbf{x}(\frac{Mn}{L}), & \text{if } \frac{Mn}{L} = k \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} (M, L) = 1 \quad &\Longleftrightarrow \quad \frac{Mn}{L} \text{ is integer } \mathit{if\ and\ only\ if}\ \frac{n}{L} \text{ is integer} \\ &\Longleftrightarrow \quad \mathbf{u}'(n) = \mathbf{u}(n) \end{aligned}$$

Therefore, $(\uparrow L)(\downarrow M)\mathbf{x} = (\downarrow M)(\uparrow L)\mathbf{x}$ *if and only if* L and M are relatively prime.
The odd-numbered components become zeros after $(\uparrow 2)(\downarrow 2)$.
In $X(z) = \sum \mathbf{x}(n)z^{-n}$, the odd-numbered coefficients are zero, therefore $X(z) = \sum \mathbf{x}(2n)z^{-2n}$.

<u>Problems 7</u>: In smoothing $u(n)$ to get the final output $w = Fu$, which filters F will interpolate and not change the even samples: $w(2k) = u(2k)$?

7. $\mathbf{f} = (...0, \frac{1}{2}, 1, \frac{1}{2}, 0...)$. This filter keeps the even samples and replaces the zero values in the odd-numbered components $\mathbf{u}(2k+1)$ with a simple "average": $\frac{\mathbf{u}(2k)+\mathbf{u}(2k+2)}{2}$. The matrix form is:

$$\begin{bmatrix} \cdot \\ \cdot \\ \mathbf{w}(-3) \\ \mathbf{w}(-2) \\ \mathbf{w}(-1) \\ \mathbf{w}(0) \\ \mathbf{w}(1) \\ \mathbf{w}(2) \\ \mathbf{w}(3) \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{1}{2} & 0 & 0 & 0 & \cdot \\ \cdot & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & \cdot \\ \cdot & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \cdot \\ \cdot & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \cdot \\ \cdot & 0 & 0 & 0 & \frac{1}{2} & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ 0 \\ \mathbf{u}(-2) \\ 0 \\ \mathbf{u}(0) \\ 0 \\ \mathbf{u}(2) \\ 0 \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \frac{\mathbf{u}(-4)+\mathbf{u}(-2)}{2} \\ \mathbf{u}(-2) \\ \frac{\mathbf{u}(-2)+\mathbf{u}(0)}{2} \\ \mathbf{u}(0) \\ \frac{\mathbf{u}(0)+\mathbf{u}(2)}{2} \\ \mathbf{u}(2) \\ \frac{\mathbf{u}(2)+\mathbf{u}(4)}{2} \\ \cdot \\ \cdot \end{bmatrix}$$

In general, a halfband filter satisfying $\sum_n h(n)$ will interpolate and not change the even samples.

4. Problem Set 4.1:

Problem 3: Find all filters if $H_0(z) = \left(\frac{1+z^{-1}}{2}\right)^3$ and $P_0(z) = \frac{1}{16}\left(-1+9z^{-2}+16z^{-3}+9z^{-4}-z^{-6}\right)$.

3. $P_0(z)$ has six roots. Four real roots correspond to the four zeros of $(1+z^{-1})^4$ at $z=-1$. The other two complex roots are $c = 2-\sqrt{3}$, $\frac{1}{c} = 2+\sqrt{3}$. Given one factor $H_0(z) = (\frac{1+z^{-1}}{2})^3$, roots c, $\frac{1}{c}$ and -1 will go into the synthesis filter $F_0(z)$. This factorization yields:

$$H_0(z) = (\frac{1+z^{-1}}{2})^3 = \frac{1}{8}(1+3z^{-1}+3z^{-2}+z^{-3})$$
$$F_0(z) = (\frac{1+z^{-1}}{2})(c-z^{-1})(\frac{1}{c}-z^{-1}) = \frac{1}{2}(-1+3z^{-1}+3z^{-2}-z^{-3})$$

For aliasing cancellation, we choose

$$H_1(z) = F_0(-z) \quad \text{and} \quad F_1(z) = -H_0(-z)$$

Therefore, the highpass filters are

$$H_1(z) = \frac{1}{2}(-1-3z^{-1}+3z^{-2}+z^{-3})$$
$$F_1(z) = \frac{1}{8}(-1+3z^{-1}-3z^{-2}+z^{-3})$$

Problems 4: If an FIR filter $H_0(z)$ has three or more coefficients, explain why $H_0^2(z)$ has at least two odd powers. Then $H_0^2(z)-H_0^2(-z) = 2z^{-l}$ is impossible. The "alternating signs" construction is not PR.

4. See the proof of Theorem 5.4 on page 159.

Problems 9: the 10th degree halfband polynomial $P_0(z) = (1+z^{-1})^6 Q(z)$ has four complex roots r, $\bar{r}$, r^{-1}, $\bar{r}^{-1}$ in the right halfplane (roots of Q). Draw a figure to show the ten roots and how Daubechies 6/6 filters will divide them: r and $\bar{r}$ are separated from r^{-1} and $\bar{r}^{-1}$.

9. $\mathbf{Q}(z) = \frac{1}{256}(3-18z^{-1}+38z^{-2}-18z^{-3}+3z^{-4})$ is chosen so that $P_0(z) = (1+z^{-1})^6\mathbf{Q}(z)$ has only one odd term z^{-5} (with coefficient 1). Then the 10th degree halfband polynomial is:

$$P_0(z) = \frac{1}{256}(3-25z^{-2}+150z^{-4}+256z^{-5}+150z^{-6}-25z^{-8}+3z^{-10})$$

$(1+z^{-1})^6$ has six zeros at $z=-1$. To find the roots of $\mathbf{Q}(z)$, we multiply $\mathbf{Q}(z)$ by z^2. Then we get a quadratic formula for $z+\frac{1}{z}$:

$$3(z+\frac{1}{z})^2 - 18(z+\frac{1}{z}) + 32 = 0$$

$$z+\frac{1}{z} = 3+\frac{\sqrt{15}}{3}i \quad \text{and} \quad z+\frac{1}{z} = 3-\frac{\sqrt{15}}{3}i$$

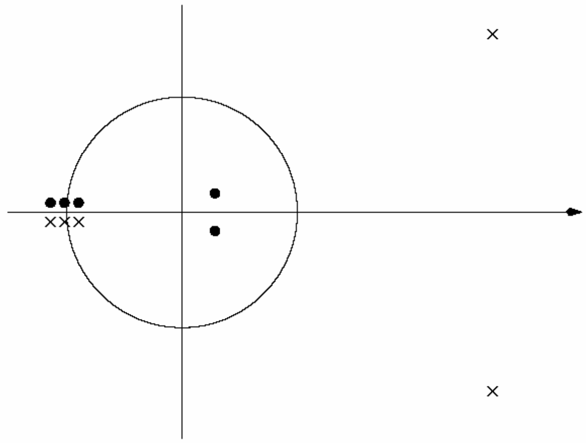
This gives another four complex roots of $P_0(z)$ (compare the solution to problem 2 in §5.5):

$$r = \frac{3-\frac{\sqrt{15}}{3}i-\sqrt{\frac{10}{3}-2\sqrt{15}i}}{2} \qquad r^{-1} = \frac{3-\frac{\sqrt{15}}{3}i+\sqrt{\frac{10}{3}-2\sqrt{15}i}}{2}$$

$$\bar{r} = \frac{3+\frac{\sqrt{15}}{3}i-\sqrt{\frac{10}{3}+2\sqrt{15}i}}{2} \qquad \bar{r}^{-1} = \frac{3+\frac{\sqrt{15}}{3}i+\sqrt{\frac{10}{3}+2\sqrt{15}i}}{2}$$

Problems 11: Find the actual 4th degree $Q(z)$ that makes $P_0(z)$ halfband. If possible compute its roots.

r and $\bar{r}$ are inside the unit circle. Their reciprocals are outside. We plot the ten roots below. Daubechies 6/6 filters put $r - z^{-1}$, $\bar{r} - z^{-1}$, $(1 + z^{-1})^3$ (the roots inside the unit circle) into $H_0(z)$ and $\frac{1}{r} - z^{-1}$, $\frac{1}{\bar{r}} - z^{-1}$, $(1 + z^{-1})^3$ into $F_0(z)$ so that the analysis filter has minimum phase. The splitting is also shown in the figure. This construction is orthogonal.



● = roots of H_0 and × = roots of F_0