Day 2

1. Problem Set 3.4:

Problems 3: Simplify the following system:

$$x(n) \longrightarrow \boxed{z^{-4}} \longrightarrow \boxed{\downarrow 2} \longrightarrow \boxed{\uparrow 3} \longrightarrow \boxed{\downarrow 2} \longrightarrow y(n)$$

What is $Y(z)$ in term of $X(z)$? Find $y(n)$ for the following inputs: $x(n)=\delta(n)$, $x(n)=(\ldots,\ 1,\ 1,\ 1,\ 1,\ \ldots)$, $x(n)=(\ldots,\ 1,\ -1,\ 1,\ -1,\ \ldots)$.

3. Since (2,3)=1, $(\uparrow 3)$ and $(\downarrow 2)$ are commutable. It leads to:

$$(\downarrow 2)(\uparrow 3)(\downarrow 2) = (\uparrow 3)(\downarrow 2)(\downarrow 2) = (\uparrow 3)(\downarrow 4)$$

For $H(z) = z^{-4} = G(z^4)$, where $G(z) = z^{-1}$ is a simple delay, we can apply the first noble identity to do the sampling first. Then the simplified system has the following block form:

$$x(n) \longrightarrow \boxed{\downarrow 4} \longrightarrow \boxed{z^{-1}} \longrightarrow \boxed{\uparrow 3} \longrightarrow y(n)$$

The z-transform of $\mathbf{v} = (\downarrow 4)\mathbf{x}$ is:

$$V(z) = \frac{1}{4}\sum_{k=0}^{3} X(z^{\frac{1}{4}} e^{\frac{2\pi k i}{4}})$$

Followed by a simple delay results:

$$U(z) = \frac{z^{-1}}{4}\sum_{k=0}^{3} X(z^{\frac{1}{4}} e^{\frac{2\pi k i}{4}})$$

Finally, upsampling by 3 produces:

$$Y(z) = U(z^3) = \frac{z^{-3}}{4}\sum_{k=0}^{3} X(z^{\frac{3}{4}} e^{\frac{2\pi k i}{4}})$$

(1) For $\mathbf{x}(n) = \delta(n)$, $\mathbf{y}(n) = \delta(n-3)$.
(2) For $\mathbf{x}(n) = (\ldots, 1, 1, 1, 1, \ldots)$, $\mathbf{y}(n) = (\ldots, 0, 1, 0, 0, 1, 0, 0, 1, 0, \ldots)$.
(3) For $\mathbf{x}(n) = (\ldots, 1, -1, 1, -1, \ldots)$, $\mathbf{y}(n) = (\ldots, 0, 1, 0, 0, 1, 0, 0, 1, 0, \ldots)$.

2. Problem Set 4.2:

Problems 1: Find $X_{even}(z)$ and $X_{odd}(z)$ when $X(z) = 1 + 2z^{-5} + z^{-10}$. Verify that $X_{even}(z^2) = \frac{1}{2}\left(X(z) + X(-z)\right)$ and $X_{odd}(z^2) = \frac{z}{2}\left(X(z) - X(-z)\right)$. The odd definition involves an advance!.

1. $X_{even}(z) = 1 + z^{-5}$, $X_{odd}(z) = 2z^{-2}$. One can easily verify that $\frac{1}{2}(X(z)+X(-z)) = 1+z^{-10} = X_{even}(z^2)$ and $\frac{z}{2}(X(z) - X(-z)) = 2z^{-4} = X_{odd}(z^2)$.

Problem 4: Polyphase Representation of an IIR Transfer function

Let $H(z)=\frac{1}{1-az^{-1}}$ where $0<a<1$. Its impulse response is $h(n)=a^n$ for $n\geq 0$ (and zero for negative n). The phases are $h_{even}(n)=(1, \quad a^2, \quad a^4, \quad ...)$ and $h_{odd}(n)=(a, \quad a^3, \quad a^5, \quad ...)$. The z-transform are $H_{even}(n)=\frac{1}{(1-a^2z^{-1})}$ and $H_{odd}(n)=\frac{a}{(1-a^2z^{-1})}$. This method is very cumbersome. One has to find the impulse response $h(n)$, then its even and odd parts $h_{even}(n)$ and $h_{odd}(n)$, then the z-transform.

An alternative method is to write $H(z)=\frac{1}{1-az^{-1}}$ directly as $H(z)=H_{even}(z^2)+z^{-1}H_{odd}(z^2)$. The dominator must be a function of z^2. So multiply above and below by $1+az^{-1}$:

$$H(z)=\frac{1}{1-az^{-1}}\frac{1+az^{-1}}{1+az^{-1}}=\frac{1+az^{-1}}{1-a^2z^{-2}}=\frac{1}{1-a^2z^{-2}}+z^{-1}\frac{a}{1-a^2z^{-2}}$$

This displays $H_{even}(z)$ and $H_{odd}(z)$. An N^{th} order filter can be factored as a cascade of first-order sections, and this method applies to each section.

(a). Let $H(z)=\frac{1}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$. Factor $H(z)$ into two first-order poles. Find the polyphase components of $H(z)$.

(b). Let $H(z)=\frac{1+2z^{-1}+5z^{-2}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$. What are this polyphase components?

4. (a) The first-order form of $\mathbf{H}(z)$ is

$$\mathbf{H}(z)=\frac{1}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}=\frac{-2}{1-\frac{1}{3}z^{-1}}+\frac{3}{1-\frac{1}{2}z^{-1}}=\hat{\mathbf{H}}(z)+\bar{\mathbf{H}}(z)$$

Applying the IIR polyphase formula for the first-order pole that have been given in the book, we rewrite $\hat{\mathbf{H}}(z)$ and $\bar{\mathbf{H}}(z)$ and find the components for $\hat{\mathbf{H}}$, $\bar{\mathbf{H}}$:

$$\begin{aligned}\hat{\mathbf{H}}(z) &= \frac{-2}{1-\frac{1}{3}z^{-1}}=\frac{-2}{1-\frac{1}{9}z^{-2}}-z^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}z^{-2}}\\ \bar{\mathbf{H}}(z) &= \frac{3}{1-\frac{1}{2}z^{-1}}=\frac{3}{1-\frac{1}{4}z^{-2}}+z^{-1}\frac{\frac{3}{2}}{1-\frac{1}{4}z^{-2}}\end{aligned}$$

Summing the even phases of $\hat{\mathbf{H}}(z)$ and $\bar{\mathbf{H}}(z)$ gives $H_{even}(z)$:

$$H_{even}(z)=\frac{-2}{1-\frac{1}{9}z^{-1}}+\frac{3}{1-\frac{1}{4}z^{-1}}$$

Summing the odd phases of $\hat{\mathbf{H}}(z)$ and $\hat{\mathbf{H}}(z)$ gives $H_{odd}(z)$:

$$H_{odd}(z)=\frac{-\frac{2}{3}}{1-\frac{1}{9}z^{-1}}+\frac{\frac{3}{2}}{1-\frac{1}{4}z^{-1}}$$

(b) Let $\mathbf{F}(z) = \frac{1}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$, $\mathbf{G}(z) = 1 + 2z^{-1} + 5z^{-2}$. Then $\mathbf{H}(z) = \mathbf{F}(z)\mathbf{G}(z)$. $\mathbf{F}$ and $\mathbf{G}$ has phase components:

$$F_{even}(z) = \frac{-2}{1-\frac{1}{9}z^{-1}} + \frac{3}{1-\frac{1}{4}z^{-1}} \qquad \text{and} \qquad F_{odd}(z) = \frac{-\frac{2}{3}}{1-\frac{1}{9}z^{-1}} + \frac{\frac{3}{2}}{1-\frac{1}{4}z^{-1}}$$

$$G_{even}(z) = 1 + 5z^{-1} \qquad \text{and} \qquad G_{odd}(z) = 2$$

The even part of $\mathbf{H}(z)$ is obtained from even times even plus odd times odd:

$$F_{even}(z^2)G_{even}(z^2) + z^{-2}F_{odd}(z^2)G_{odd}(z^2)$$

The odd part of $\mathbf{H}(z)$ is even times odd plus odd times even (with a delay):

$$z^{-1}(F_{even}(z^2)G_{odd}(z^2) + F_{odd}(z^2)G_{even}(z^2))$$

After downsampling, z^{-2} becomes z^{-1}. We find the polyphase components for $\mathbf{H}$:

$$\begin{aligned} H_{even}(z) &= F_{even}(z)G_{even}(z) + z^{-1}F_{odd}(z)G_{odd}(z) \\ &= \frac{-2-\frac{34}{3}z^{-1}}{1-\frac{1}{9}z^{-1}} + \frac{3+18z^{-1}}{1-\frac{1}{4}z^{-1}} \end{aligned}$$

$$\begin{aligned} H_{odd}(z) &= F_{even}(z)G_{odd}(z) + F_{odd}(z)G_{even}(z) \\ &= \frac{-\frac{14}{3}-\frac{10}{3}z^{-1}}{1-\frac{1}{9}z^{-1}} + \frac{\frac{15}{2}+\frac{15}{2}z^{-1}}{1-\frac{1}{4}z^{-1}} \end{aligned}$$

Problem 7: Let $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 4z^{-4} + 3z^{-5} + 2z^{-6} + z^{-7}$. Find the polyphase components $H_{even}(z)$ and $H_{odd}(z)$ for antisymmetric filters of even length and symmetric filters of odd length?

7. The polyphase components for $\mathbf{H}$ are:

$$\begin{aligned} H_{even}(z) &= 1 + 3z^{-1} + 4z^{-2} + 2z^{-3} \\ H_{odd}(z) &= 2 + 4z^{-1} + 3z^{-2} + z^{-3} \end{aligned}$$

For anti-symmetric filters of even length, $h_{odd}(n)$ is the flip of $h_{even}(n)$ with a minus sign. In z-transform domain, it can be expressed as

$$H_{even}(z) = -z^{\frac{N-1}{2}}H_{odd}(z^{-1})$$

For symmetric filters of odd length, since the symmetric pair $\mathbf{h}(k) = \mathbf{h}(n-k)$ go together into the same phase, there is no direct relation between $H_{even}(z)$ and $H_{odd}(z)$. But the general identity holds for all: $\mathbf{H}(z) = H_{even}(z^2) + z^{-1}H_{odd}(z)$.

3. Problem Set 4.3:

Problem 2: If $H_m^T(z^{-1})H_m(z) = 2I$ show from

$$\begin{bmatrix} H_{0,even}(z^2) & H_{0,odd}(z^2) \\ H_{1,even}(z^2) & H_{1,odd}(z^2) \end{bmatrix}\begin{bmatrix} 1 & \\ & z^{-1} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} H_0(z) & H_0(z) \\ H_1(z) & H_1(z) \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

that $H_p^T(z^{-1})H_p(z) = I$

2. The polyphase matrix is related to the modulation matrix as (shown by Eqn. 4.48):

$$\mathbf{H_p}(z) = \frac{1}{2}\mathbf{H_m}(z^{\frac{1}{2}}) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & z^{\frac{1}{2}} \end{bmatrix}$$

By transposing the matrices (in reverse order) and inverting z, the last expression becomes

$$\mathbf{H_p}^T(z^{-1}) = \frac{1}{2} \begin{bmatrix} 1 & \\ & z^{-\frac{1}{2}} \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^T \mathbf{H_m}^T(z^{-\frac{1}{2}})$$

Given $\mathbf{H_m}^T(z^{-1})\mathbf{H_m}(z) = 2\mathbf{I}$, then

$$\begin{aligned}
\mathbf{H_p}^T(z^{-1})\mathbf{H_p}(z) &= \frac{1}{4} \begin{bmatrix} 1 & \\ & z^{-\frac{1}{2}} \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^T \mathbf{H_m}^T(z^{-\frac{1}{2}})\mathbf{H_m}(z^{\frac{1}{2}}) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & z^{\frac{1}{2}} \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} 1 & \\ & z^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} 2\mathbf{I} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & z^{\frac{1}{2}} \end{bmatrix} \\
&= \begin{bmatrix} 1 & \\ & z^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 1 & \\ & z^{\frac{1}{2}} \end{bmatrix} \\
&= \mathbf{I}
\end{aligned}$$

Problem 17: Find the analysis filters $H_0(z)$ and $H_1(z)$ for the following matrices:

(b) $H_p(z) = \begin{bmatrix} 1+2z^{-2} & 1+z^{-1} \\ 1-z^{-1} & 2+z^{-1} \end{bmatrix} \begin{bmatrix} 2 & 1+z^{-2} \\ 1-z^{-1} & -z^{-3} \end{bmatrix}$.

(c) $H_p(z) = \begin{bmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{bmatrix}$.

17. Assume $H_{00}(z)$, $H_{01}(z)$, $H_{10}(z)$ and $H_{11}(z)$ are the four entries in $\mathbf{H_p}(z)$:

$$\mathbf{H_p}(z) = \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \end{bmatrix}$$

then

$$\begin{aligned}
H_0(z) &= H_{00}(z^2) + z^{-1}H_{01}(z^2) \\
H_1(z) &= H_{10}(z^2) + z^{-1}H_{11}(z^2)
\end{aligned}$$

(a) Two analysis filters can be obtained directly from $\mathbf{H_p}(z)$:

$$\begin{aligned}
H_0(z) &= 1 + 2z^{-1} + 2z^{-2} - z^{-3} - z^{-4} \\
H_1(z) &= z^{-1} + 2z^{-3} + z^{-5} + z^{-6}
\end{aligned}$$

(b) The cascade of two FIR filters is

$$\mathbf{H_p}(z) = \begin{bmatrix} 3+3z^{-2} & 1+3z^{-2}-z^{-3}+z^{-4} \\ 4-3z^{-1}-z^{-2} & 1-z^{-1}+z^{-2}-3z^{-3}-z^{-4} \end{bmatrix}$$

$$\Longrightarrow \quad H_0(z) = 3 + z^{-1} + 3z^{-4} + 3z^{-5} - z^{-7} + z^{-9}$$

$$H_1(z) = 4 + z^{-1} - 3z^{-2} - z^{-3} - z^{-4} + z^{-5} - 3z^{-7} - z^{-9}$$

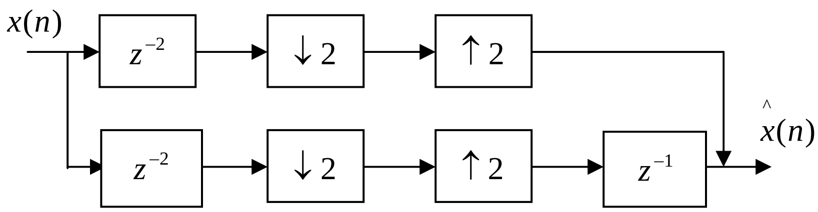
(c) Lattice cascade results in

$$\mathbf{H_p}(z) = \begin{bmatrix} c_1c_2 - s_1s_2z^{-1} & c_1s_2 + c_2s_1z^{-1} \\ -c_2s_1 - c_1s_2z^{-1} & -s_1s_2 + c_1c_2z^{-1} \end{bmatrix}$$

$$\Longrightarrow \quad H_0(z) = c_1c_2 + c_1s_2z^{-1} - s_1s_2z^{-2} + c_2s_1z^{-3}$$
$$H_1(z) = -c_2s_1 - s_1s_2z^{-1} - c_1s_2z^{-2} + c_1c_2z^{-3}$$

4. Problem Set 4.4:

Problem 9: Find the matrices $H_p(z)$ and $F_p(z)$. Is the system PR?



9. The analysis bank has $H_0(z) = z^{-2}$ and $H_1(z) = z^{-1}$. This leads to the Type 1 polyphase matrix:

$$\mathbf{H_p}(z) = \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

The corresponding Type 2 synthesis matrix for $F_0(z) = 1$ and $F_1(z) = z^{-1}$ is:

$$\mathbf{F_p}(z) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The output from the lowpass channel is the even components (delayed by 2):

$$w_0(2k) = \mathbf{x}(2k-2) \quad \text{and} \quad w_0(2k+1) = 0$$

The highpass channel keeps only the odd components and delays the signal by 2:

$$w_1(2k) = 0 \quad \text{and} \quad w_1(2k+1) = \mathbf{x}(2k-1)$$

Therefore, the sum of w_0 and w_1 reconstructs $\hat{\mathbf{x}}(n) = \mathbf{x}(n-2)$. This is a PR system.

5. Problem Set 5.2:

Problem 2: For any four coefficients $h(0),\ldots,h(3)$, verify that

$$\left|H_{even}(z)\right|^2 + \left|H_{odd}(z)\right|^2 = \frac{1}{2}\left(\left|H(z)\right|^2 + \left|H(-z)\right|^2\right)$$

Then condition O for the polyphase equals Condition O for modulation.

2. $|H_{even}(z^2)|^2 + |H_{odd}(z^2)|^2 = \frac{1}{2}(|H(z)|^2 + |H(-z)|^2)$ for a four-tap filter: Note: Use $H_{even}(z^2)$ instead of $H_{even}(z)$ as the book suggests. Same for $H_{odd}(z^2)$

$$\begin{aligned}
\text{LHS} &= |H_{even}(z^2)|^2 + |H_{odd}(z^2)|^2 \\
&= (\mathbf{h}(0) + \mathbf{h}(2)z^{-2})(\mathbf{h}(0) + \mathbf{h}(2)z^2) + (\mathbf{h}(1) + \mathbf{h}(3)z^{-2})(\mathbf{h}(0) + \mathbf{h}(2)z^2) \\
&= \mathbf{h}^2(0) + \mathbf{h}^2(1) + \mathbf{h}^2(2) + \mathbf{h}^2(3) + (\mathbf{h}(0)\mathbf{h}(2) + \mathbf{h}(1)\mathbf{h}(3))(z^{-2} + z^2) \\
\text{RHS} &= \frac{1}{2}(|H(z)|^2 + |H(-z)|^2) \\
&= \frac{1}{2}\Big((\mathbf{h}(0) + \mathbf{h}(1)z^{-1} + \mathbf{h}(2)z^{-2} + \mathbf{h}(3)z^{-3})(\mathbf{h}(0) + \mathbf{h}(1)z + \mathbf{h}(2)z^2 + \mathbf{h}(3)z^3) + \\
&\qquad (\mathbf{h}(0) - \mathbf{h}(1)z^{-1} + \mathbf{h}(2)z^{-2} - \mathbf{h}(3)z^{-3})\mathbf{h}(0) - \mathbf{h}(1)z + \mathbf{h}(2)z^2 - \mathbf{h}(3)z^3)\Big) \\
&= \mathbf{h}^2(0) + \mathbf{h}^2(1) + \mathbf{h}^2(2) + \mathbf{h}^2(3) + (\mathbf{h}(0)\mathbf{h}(2) + \mathbf{h}(1)\mathbf{h}(3))(z^{-2} + z^2)
\end{aligned}$$

Problem 4: Find d by alternating flip of $c = (c(0),...,c(5))$. Verify equation $\sum c(n)d(n-2k) = 0$ directly to show that c is double-shift orthogonal to d.

4. Let $\mathbf{c} = (\mathbf{c}(0), \mathbf{c}(1), \mathbf{c}(2), \mathbf{c}(3), \mathbf{c}(4), \mathbf{c}(5))$, then

$$\mathbf{d}(k) = (-1)^k \mathbf{c}(5-k) = (\mathbf{c}(5), -\mathbf{c}(4), \mathbf{c}(3), -\mathbf{c}(2), \mathbf{c}(1), -\mathbf{c}(0))$$

$\mathbf{d}(k)$ is double-shift orthogonal to $\mathbf{c}(k)$:

$$\sum \mathbf{c}(k)\mathbf{d}(k) = \mathbf{c}(0)\mathbf{c}(5) - \mathbf{c}(1)\mathbf{c}(4) + \mathbf{c}(2)\mathbf{c}(3) - \mathbf{c}(3)\mathbf{c}(2) + \mathbf{c}(4)\mathbf{c}(1) - \mathbf{c}(5)\mathbf{c}(0) = 0$$

$$\sum \mathbf{c}(k)\mathbf{d}(k-2) = \mathbf{c}(0)\mathbf{c}(3) - \mathbf{c}(1)\mathbf{c}(2) + \mathbf{c}(2)\mathbf{c}(1) - \mathbf{c}(3)\mathbf{c}(0) = 0$$

$$\sum \mathbf{c}(k)\mathbf{d}(k-4) = \mathbf{c}(0)\mathbf{c}(1) - \mathbf{c}(1)\mathbf{c}(0) = 0$$

6. Problem Set 5.4:

Problem 3: Why must all roots of $P(z)$ on the unit circle have even multiplicity, to allow $P(z) = C(z)C(z^{-1})$ and $P(\omega) = |C(\omega)|^2$?

3. Let z_k be a factor of $P(z)$ such that $|z_k| = 1$. To factor $P(z)$ into polynomials with real coefficients, z_k and $\overline{z_k}$ must go together into one factor, say $C(z)$. This constrains the other factor, $C(z^{-1})$, to have roots at z_k^{-1} and $\overline{z_k^{-1}}$, but because of the relation $z_k^{-1} = \overline{z_k}$ for any root on the unit circle, we conclude that $C(z^{-1})$ must also have roots at z_k and $\overline{z_k}$. Hence, any root on the unit circle must have even multiplicity.