Day 3

1. Problem Set 6.1:

Problem 1: Explain why the scaling requirement, that $f(t)$ is in V_j if and only if $f(2t)$ is in V_{j+1}, can be restated as $\hat{f}(\omega)$ is in $\hat{V}_j$ if and only if $\hat{f}(2\omega)$ is in $\hat{V}_{j-1}$. Here $\hat{V}_j$ is the space of Fourier transforms of functions in V_j.

1. The Fourier transform of $f(2t)$ is $\frac{1}{2}\hat{f}(\frac{\omega}{2})$:

$$\int_{-\infty}^{\infty} f(2t)e^{-i\omega t}dt = \frac{1}{2}\int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau/2}d\tau = \frac{1}{2}\hat{f}(\frac{\omega}{2})$$

Therefore $\hat{f}(\frac{\omega}{2})$ is in $\hat{V}_{j+1}$ (the transform of functions in V_{j+1}) when $\hat{f}(\omega)$ is in $\hat{V}_j$. This means that $\hat{f}(\omega)$ is in $\hat{V}_j$ when $\hat{f}(2\omega)$ is in $\hat{V}_{j-1}$.

Problem 2: Find 2 by 2 matrices $c(0)$ and $c(1)$ so that the box function $\phi_1(t)$ and sloping line $\phi_2(t) = 1 - 2t$ satisfy

$$\begin{bmatrix}\phi_1(t)\\ \phi_2(t)\end{bmatrix} = c(0)\begin{bmatrix}\phi_1(2t)\\ \phi_2(2t)\end{bmatrix} + c(1)\begin{bmatrix}\phi_1(2t-1)\\ \phi_2(2t-1)\end{bmatrix}.$$

2. The problem is to show that the only compactly supported $\phi(t)$ whose translates are a basis for V_0 is a box function (possibly shifted). The equivalent statement is that the only FIR filter with FIR inverse is a delay (a shift). Another equivalent statement: $P(z)$ and $1/P(z)$ are both polynomials only if $P(z)$ has only one term (a monomial).
In this problem $B(t) \in V_0$ so the basis of translates must give

$$B(t) = \sum_k \mathbf{a}(k)\phi(t-k)$$

At the same time the function $\phi(t) \in V_0$ and the box function gives a basis:

$$\phi(t) = \sum_k \mathbf{b}(k)B(t-k)$$

Take Fourier transform of these equations:

$$\hat{B}(\omega) = (\sum \mathbf{a}(k)e^{-i\omega k})\hat{\phi}(\omega) \quad \text{and} \quad \hat{\phi}(\omega) = (\sum \mathbf{b}(k)e^{-i\omega k})\hat{B}(\omega)$$

Thus $(\sum \mathbf{a}(k)e^{-i\omega k})(\sum \mathbf{b}(k)e^{-i\omega k}) = 1$. If these are both polynomials then they are both monomials (one nonzero coefficient). Therefore $\phi(t) = aB(t-l) =$ scaled and shifted box.

2. Problem Set 6.2:

Problem 5: Suppose the filter coefficients $h(k)$ are $\frac{1}{2}, \quad 0, \quad 0, \quad \frac{1}{2}$. Starting from the box function, take one step of the cascade algorithm and draw $\phi^{(1)}(t)$. Then take the second step and draw $\phi^{(2)}(t)$. Describe $\phi^{(i)}(t)$ - on what fraction of the interval $[0, \quad 3]$ does $\phi^{(i)}(t) = 1$?

5. See example 7.2 on page 235.

Problem 6: Suppose the only filter coefficient is $h(0)=1$. Starting from the box function $\phi^{(0)}(t)$, draw the graphs of $\phi^{(1)}(t)$ and $\phi^{(2)}(t)$. In what sense does $\phi^{(i)}(t)$ converse to the delta function $\delta(t)$? To verify the dilation equation $\delta(t)=2\delta(2t)$, multiply by the any smooth $f(t)$ and compare the integrals of both sides.

6. The cascade algorithm converges weakly to the *delta* function. The dilation equation $\delta(t) = 2\delta(2t)$ is verified by integrating $\delta(t)$ times a smooth function $f(t)$ in L^2:

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0) = \int_{-\infty}^{\infty} f(t)\delta(2t)2dt$$

3. Problem Set 6.3:

Problem 3: Show that the convolution $\phi_1(t)*\phi_2(t)$ does satisfy a dilation equation with the coefficients from $h_1 * h_2$.

3. This question is answered by Lemma 7.2 on page 253. We give another answer here (in the time domain).

To derive the dilation equation for $\Phi(t) = \phi_1(t) * \phi_2(t)$, we use $\phi_1(t) = \sum 2\mathbf{h}_1(k)\phi_1(2t-k)$ and $\phi_2(t) = \sum 2\mathbf{h}_2(k)\phi_2(2t-k)$ to evaluate the convolution of $\phi_1(t)$ with $\phi_2(t)$:

$$\begin{aligned} \Phi(t) = \phi_1(t) * \phi_2(t) &= \int_{-\infty}^{\infty} \phi_1(t-x)\phi_2(x)dx \\ &= 4\int_{-\infty}^{\infty} (\sum_i \mathbf{h}_1(i)\phi_1(2t-2x-i))(\sum_k \mathbf{h}_2(k)\phi_2(2x-k))dx \end{aligned}$$

If we interchange the order of summation and integration,

$$\Phi(t) = 4\sum_i\sum_k \mathbf{h}_1(i)\mathbf{h}_2(k)\int_{-\infty}^{\infty} \phi_1(2t-2x-i)\phi_2(2x-k)dx$$

By making substitutions $y = 2x-k$ and $l = i+k$, the last expression becomes

$$4\sum_l\sum_k \mathbf{h}_1(l-k)\mathbf{h}_2(k)\int_{-\infty}^{\infty} \phi_1(2t-y-l)\phi_2(y)\frac{dy}{2}$$

4. Problem Set 6.4:

Problem 4: If $H(\omega)$ has p zeros at $\omega=\pi$, show that $\hat{\phi}(\omega)$ has p zeros at $\omega=2n\pi$ for each $n\neq 0$.

4. See the proof of Eqn. (7.25) on page 230.